

GLAUBER DYNAMICS FOR THE QUANTUM ISING MODEL IN A TRANSVERSE FIELD ON A REGULAR TREE

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ABSTRACT. Motivated by a recent use of Glauber dynamics for Monte-Carlo simulations of path integral representation of quantum spin models [4], we analyse a natural Glauber dynamics for the quantum Ising model with a transverse field on a finite graph G . We establish strict monotonicity properties of the equilibrium distribution and we extend (and improve) the censoring inequality of Peres and Winkler to the quantum setting. Then we consider the case when G is a regular b -ary tree and prove the same fast mixing results established in [7] for the classical Ising model. Our main tool is an inductive relation between conditional marginals (known as the “cavity equation”) together with sharp bounds on the operator norm of the derivative at the stable fixed point. It is here that the main difference between the quantum and the classical case appear, as the cavity equation is formulated here in an infinite dimensional vector space, whereas in the classical case marginals belong to a one-dimensional space.

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1. INTRODUCTION

In the last years there has been a somewhat intense research in probabilistic representation of quantum spin systems with very interesting progresses exploiting stochastic geometry methods. We refer the interested reader to [2] and references therein. More recently, physicists started to consider quantum Heat Bath algorithms applied to the imaginary time path integral representation of e.g. the quantum Ising model in a transverse field (see [4] and references therein). From a mathematical point of view that accounts to introduce a so-called Glauber dynamics for the dynamical variables appearing in the path integral representation of the model, a subject that, in the context of *classical* spin models, has been thoroughly investigated in the mathematical literature in the past fifteen years because of its many different facets and connections with different areas in mathematics, physics and theoretical computer science. In the quantum setting very little is known rigorously and we are only aware of a recent contribution [1] where, for a particular choice of the Glauber chain, exponentially fast approach to the imaginary time equilibrium measure is proved via a kind of Bochner-Bakry-Emery method in a high temperature regime.

Here we investigate the simple heat bath dynamics for the imaginary time path integral representation of the quantum Ising model with a transverse field on a finite connected graph G . We first focus on properties of the single site measure and establish key *monotonicity* properties as a function of the longitudinal magnetic field and of the boundary conditions. A remarkable consequence of monotonicity is the validity of the so-called Peres-Winkler censoring inequality [8] which we re-establish and extend to the quantum case. Then, motivated by the analysis of the same model made by physicists on the Bethe lattice [4], we specialize to the case when the

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graph G is a regular b -ary tree and we succeed in extending to the quantum case much of the analysis made in [7] for the classical setting. More precisely we prove that, for any boundary condition τ , there exists exponents $\kappa = \kappa(\tau)$, $\gamma < 1$ such that, if $\kappa\gamma b < 1$, then the Glauber chain on a finite subtree of depth ℓ with boundary condition τ at the leaves, has a spectral gap uniformly positive in ℓ . In [7] the same uniformity was proved for the logarithmic Sobolev constant and, as a consequence, the mixing time could not grow faster than linear in ℓ . In the quantum setting the logarithmic Sobolev constant is easily seen to be infinite. Nevertheless, using our extension of censoring and always assuming $\kappa\gamma b < 1$, we prove that the mixing time grows linearly in ℓ . Finally, in complete analogy with the fast mixing result inside the pure plus phase for the classical Ising model, we show that, if the boundary condition is the homogeneous *plus* configuration then the key exponent κ satisfies $\kappa \leq 1/b$ for all values of the thermodynamic parameters. Therefore, in this case, $\kappa\gamma b < 1$. The same holds for *all* boundary conditions when the thermodynamic parameters are in the uniqueness regime.

All the main technical efforts in the paper are concentrated on bounding the exponent κ and it is at this point that the analysis of the quantum case becomes much more involved than its classical analog. Computing the exponent κ requires in fact a detailed study of the quantum cavity equation for the one site conditional marginal distributions and, in particular, of the derivative operator at the fixed point corresponding to the “plus” phase. In the classical case such marginals are parametrized by just one number and therefore the corresponding cavity equation is just a recursive non-linear equation in \mathbf{R} . In the quantum case instead the cavity equation becomes a recursive functional equation in a infinite dimensional vector space.

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2. THE QUANTUM ISING MODEL

2.1. The matrix formulation. We consider a finite graph $G = (V, E)$ with vertex set V and edge set E . A classical spin configuration on V is $\sigma = (\sigma_i)_{i \in V} \in \{\pm 1\}^V$. Now we define \mathcal{H} , a

$2^{|V|}$ -dimensional Hilbert space, by giving the orthonormal basis $(|\sigma\rangle)_{\sigma \in \{\pm 1\}^V}$. We also denote $\langle \sigma |$ the transpose of any basis vector $|\sigma\rangle$. Given $i \in V$, we define two linear operators on \mathcal{H} by their action on the orthonormal basis:

$$\begin{aligned}\sigma_i^z |\sigma\rangle &= \sigma_i |\sigma\rangle \\ \sigma_i^x |\sigma\rangle &= |\sigma^i\rangle\end{aligned}$$

where σ^i is the same spin configuration as σ , except for the spin at i which is flipped. Given a set of parameters β, h, λ ($\beta > 0$ is the inverse temperature, $h \in \mathbf{R}$ the longitudinal and $\lambda \geq 0$ the transverse field) we define the Hamiltonian operator

$$H = - \sum_{e=\{i,j\} \in E} \sigma_i^z \sigma_j^z - h \sum_{i \in V} \sigma_i^z - \lambda \sum_{i \in V} \sigma_i^x$$

and define the average of an observable O (a linear operator on \mathcal{H}) at inverse temperature β as

$$\langle O \rangle_{\beta, h, \lambda} = \frac{\text{Tr}(O e^{-\beta H})}{\text{Tr}(e^{-\beta H})}.$$

2.2. The path-integral representation of the quantum Ising model. We introduce first a reference measure $\varphi_{I, \lambda}$ that corresponds to a single free spin on some interval $I \subset \mathbf{R}$. A single spin is a cadlag function $\sigma : I \rightarrow \{\pm 1\}$. We call Σ_I the set of single spins, and endow it with the usual Skorohod topology. Note that two spin configurations are close when they have the same initial value, same number of flips and close time location for the flips. This set is a Polish space as the set of spin configurations with jumps at fractional times is countable and dense in Σ . The corresponding σ -algebra is generated by the events $\{\sigma(t) = +\}$, for $t \in I$. We will also consider the L^1 -distance

$$\|\sigma - \sigma'\|_1 = \int_I |\sigma(t) - \sigma'(t)| dt.$$

between two spin configurations. Now we define the distribution $\varphi_{I, \lambda}$ as follows: we consider a Poisson point process with intensity λ on I , and define σ , with equal probability, as one of the two configurations that flip exactly at the points of the Poisson process.

Now we introduce spin interactions and external fields. Given a bounded interval I and two bounded functions $\sigma, \tau : I \rightarrow \mathbf{R}$ defined Lebesgue a.s., we call

$$\sigma \bullet_I \tau = \int_I \sigma(t) \tau(t) dt. \quad (2.1)$$

When I is clear from the context (usually $I = [0, \beta]$), we drop it from the notation. Note that the product defined by (2.1) is additive in the sense that, for I_1, I_2 disjoint,

$$\sigma \bullet_{I_1 \cup I_2} \tau = \sigma \bullet_{I_1} \tau + \sigma \bullet_{I_2} \tau.$$

A spin configuration is $\sigma : V \times I \rightarrow \{\pm 1\}$ such that $\sigma_i \in \Sigma_I$, for all $i \in V$. Consider some $\beta > 0$ (inverse temperature), $h : V \times [0, \beta] \rightarrow \mathbf{R}$ integrable (the longitudinal field), $\lambda \geq 0$ (the transverse field). We define two probability measures on the set of spins configurations:

$$\begin{aligned}\mu_{G; \beta, h, \lambda}(d\sigma) &= \frac{1}{Z_{G; \beta, h, \lambda}} \exp \left(\sum_{i \sim j} \sigma_i \bullet \sigma_j + \sum_i h_i \bullet \sigma_i \right) \prod_{i \in V} \varphi_{[0, \beta], \lambda}(d\sigma_i) \\ \mu_{G; \beta, h, \lambda}^{\text{per}}(d\sigma) &= \frac{\mathbf{1}_{\{\sigma(0) = \sigma(\beta)\}}}{Z_{G; \beta, h, \lambda}^{\text{per}}} \exp \left(\sum_{i \sim j} \sigma_i \bullet \sigma_j + \sum_i h_i \bullet \sigma_i \right) \prod_{i \in V} \varphi_{[0, \beta], \lambda}(d\sigma_i)\end{aligned}$$

where $Z_{G;\beta,\mathbf{h},\lambda}$ and $Z_{G;\beta,\mathbf{h},\lambda}^{\text{per}}$ are the appropriate normalization constant that turn $\mu_{G;\beta,\mathbf{h},\lambda}$ and $\mu_{G;\beta,\mathbf{h},\lambda}^{\text{per}}$ into probability measures on the set of spin configurations. They exist as the argument of the exponential is trivially bounded by $\beta|E| + \beta\|\mathbf{h}\|_1|V|$. The boundary condition $\sigma(0) = \sigma(\beta)$ is called *periodic imaginary time boundary condition*. Note that we could also allow $\lambda : V \times I \rightarrow \mathbf{R}^+$.

The above measure are often approximated, in the physics literature, by a discrete Ising model with vertex set $V \times \{0, \dots, N\}$ and edge set $E \times \{0, \dots, N\} \cup V \times \{(0, 1), \dots, (N-1, N), (N, 0)\}$. This is usually referred to as a consequence of the Suzuki-Trotter formula. The strength of interaction is different for transversal edges than for longitudinal edges. Although we do not use this approximation (which corresponds to enabling the discontinuities only at times $\beta k/N$), it explains that the above two measures share most of the properties of discrete Ising models.

Proposition 2.1. [DLR Equation]. *Let $I = [\beta_1, \beta_2] \subset [0, \beta]$ be an interval and $A \subset V$. Let $\sigma \sim \mu_{G;\beta,\mathbf{h},\lambda}$. Conditionally on $\sigma|_{(A \times I)^c}$, the distribution of $\sigma|_{A \times I}$ is the quantum Ising model $\mu_{(A,E);I,\mathbf{h}',\lambda}(\cdot | \sigma_A(\beta_1), \sigma_A(\beta_2))$ with external field $\mathbf{h}' = \mathbf{h} + \sum_{i \in A, j \notin A} \sigma_j \delta_i$.*

Proof. This is a consequence of the independence of the restrictions of the Poisson point process to different regions together with the additivity of the product (2.1). \square

Proposition 2.2. *Let $\beta > 0$, $h \in \mathbf{R}$ and $\lambda \geq 0$. Then, for any observable O ,*

$$\langle O \rangle_{\beta,h,\lambda} = \frac{\mu_{V;\beta,h,\lambda}(\langle \sigma(\beta) | O | \sigma(0) \rangle)}{\mu_{V;\beta,h,\lambda}(\sigma(0) = \sigma(\beta))} \quad (2.2)$$

while, for any diagonal observable O ,

$$\langle O \rangle_{\beta,h,\lambda} = \mu_{V;\beta,h,\lambda}^{\text{per}}(\langle \sigma(\beta) | O | \sigma(0) \rangle). \quad (2.3)$$

Proof. We define a matrix $W_{\beta,h,\lambda}$ by the prescription of its coordinates

$$\langle \sigma | W_{\beta,h,\lambda} | \sigma' \rangle = 2^{|V|} \int \exp \left(\sum_{i \sim j} \sigma_i \bullet \sigma_j + h \sum_i \mathbf{1} \bullet \sigma_i \right) \mathbf{1}_{\{\sigma(0) = \sigma, \sigma(\beta) = \sigma'\}} \prod_{i \in V} d\varphi_{[0,\beta],\lambda}(\sigma_i).$$

Once we prove that

$$W_{\beta,h,\lambda} = e^{-\beta\lambda|V|} \times \exp(-\beta H) \quad (2.4)$$

the claim follows from the definition of the average of an observable:

$$\begin{aligned} \langle O \rangle_{\beta,h,\lambda} &= \frac{\text{Tr}(O e^{-\beta H})}{\text{Tr}(e^{-\beta H})} = \frac{\sum_{\sigma} \langle \sigma | O e^{-\beta H} | \sigma \rangle}{\sum_{\sigma} \langle \sigma | e^{-\beta H} | \sigma \rangle} \\ &= \frac{\sum_{\sigma, \sigma'} \langle \sigma | O | \sigma' \rangle \langle \sigma' | e^{-\beta H} | \sigma \rangle}{\sum_{\sigma} \langle \sigma | e^{-\beta H} | \sigma \rangle} \end{aligned}$$

as

$$\begin{aligned} \langle \sigma' | e^{-\beta H} | \sigma \rangle &= e^{\beta\lambda|V|} \langle \sigma' | W_{\beta,h,\lambda} | \sigma \rangle \\ &= e^{\beta\lambda|V|} 2^{|V|} Z_{V;\beta,h,\lambda} \mu_{V;\beta,h,\lambda}(\sigma(0) = \sigma', \sigma(\beta) = \sigma). \end{aligned}$$

So we focus now on the proof of (2.4). According to the additivity of the product (2.1) and to the independence of the restriction of the Poisson process to $[0, \beta_1]$ and $[\beta_1, \beta_2]$, we have

$$W_{\beta_1+\beta_2,h,\lambda} = W_{\beta_1,h,\lambda} \times W_{\beta_2,h,\lambda}, \forall \beta_1, \beta_2 \geq 0. \quad (2.5)$$

Now we compute the asymptotic of $\langle \sigma | W_{\beta,h,\lambda} | \sigma' \rangle$ when $\beta \rightarrow 0$. The probability that the Poisson point process has two points or more on $V \times [0, \beta]$ is $O(\beta^2)$, therefore when σ and σ' differ at two

points or more, $\langle \sigma | W_{\beta, h, \lambda} | \sigma' \rangle = O(\beta^2)$. When $\sigma' = \sigma^i$, the probability that σ_i is discontinuous is $\beta\lambda + O(\beta^2)$, therefore $\langle \sigma | W_{\beta, h, \lambda} | \sigma' \rangle = \beta\lambda + O(\beta^2) = \beta\langle \sigma | H | \sigma' \rangle + O(\beta^2)$. Finally, when $\sigma' = \sigma$, up to $O(\beta^2)$ only the configurations with no flips (probability $\exp(-\beta\lambda|V|)$) contribute to $\langle \sigma | W_{\beta, h, \lambda} | \sigma' \rangle$, therefore in that case

$$\begin{aligned} \langle \sigma | W_{\beta, h, \lambda} | \sigma' \rangle &= \exp \left(\beta \left(\sum_{i \sim j} \sigma_i \sigma_j + h \sum_i \sigma_i \right) - \beta\lambda|V| \right) + O(\beta^2) \\ &= 1 - \beta\langle \sigma | H | \sigma' \rangle - \beta\lambda|V| + O(\beta^2). \end{aligned}$$

In other words we have shown that

$$W_{\beta, h, \lambda} = (1 - \beta\lambda|V|)I - \beta H + O(\beta^2) \quad (2.6)$$

as $\beta \rightarrow 0$, where I is the matrix that corresponds to the identity of \mathcal{H} . Combining (2.5) and (2.6) we obtain, for any $\beta \geq 0$ and any σ' ,

$$\begin{aligned} \frac{d}{d\beta} W_{\beta, h, \lambda} | \sigma' \rangle &= \left[\frac{d}{d\beta'} W_{\beta', h, \lambda} \right]_{\beta'=0} \times W_{\beta, h, \lambda} | \sigma' \rangle \\ &= (-\lambda|V|I - H) \times W_{\beta, h, \lambda} | \sigma' \rangle. \end{aligned}$$

This proves (2.4) as $\exp(-\beta\lambda|V|I - \beta H)$ is the unique solution to this differential equation. \square

3. THE SINGLE SITE SPIN MEASURE

In all this section we assume that $V = \{x\}$ is a single site. Most of the time β, λ and sometimes h will be from the context and they will not appear in the notation. We state several important facts about this single site measure. In what follows $\|\cdot\|_{TV}$ denotes the usual variation distance (see also (5.4)).

Proposition 3.1. *Let $\beta, M < \infty$. There exists $\gamma < 1$ and $\Gamma = 1/\log(3)$ such that, for any h, h', λ such that $\|h\|_1, \|h'\|_1, \lambda < M$, for any $\pi \in \{\emptyset, \text{per}\}$, the following inequalities holds*

$$\|\mu_{h'}^\pi - \mu_h^\pi\|_{TV} \leq \gamma \quad (3.1)$$

$$\|\mu_{h'}^\pi - \mu_h^\pi\|_{TV} \leq \Gamma \|h' - h\|_1. \quad (3.2)$$

Proof. Inequality (3.1) is an immediate consequence of the fact that, under our assumptions, $\mu_h(\{+\})$ is uniformly bounded from below. For (3.2) we introduce φ^π , the single spin free measure conditioned on the imaginary time boundary condition π and compute

$$\begin{aligned} \|\mu_{h'}^\pi - \mu_h^\pi\|_{TV} &= \frac{1}{2} \int \left| \frac{e^{h' \bullet \sigma}}{\int e^{h' \bullet \sigma'} d\varphi^\pi(\sigma')} - \frac{e^{h \bullet \sigma}}{\int e^{h \bullet \sigma'} d\varphi^\pi(\sigma')} \right| d\varphi^\pi(\sigma) \\ &\leq \frac{1}{2} \frac{\int |e^{h' \bullet \sigma + h \bullet \sigma'} - e^{h \bullet \sigma + h' \bullet \sigma'}| d\varphi^\pi(\sigma) d\varphi^\pi(\sigma')}{\int e^{h \bullet \sigma + h' \bullet \sigma'} d\varphi^\pi(\sigma) d\varphi^\pi(\sigma')} \\ &= \frac{1}{2} \frac{\int e^{h \bullet \sigma + h' \bullet \sigma'} |e^{(h' - h) \bullet (\sigma - \sigma')} - 1| d\varphi^\pi(\sigma) d\varphi^\pi(\sigma')}{\int e^{h \bullet \sigma + h' \bullet \sigma'} d\varphi^\pi(\sigma) d\varphi^\pi(\sigma')} \\ &\leq \frac{\exp(\|h' - h\|_1) - 1}{2}. \end{aligned}$$

The claim follows from the inequality $(e^x - 1)/2 \leq \Gamma x$, for every $x > 0$ such that $\Gamma x \leq 1$. \square

The second property is *strict monotonicity*, a crucial fact for our analysis of the dynamics. Let $f : \Sigma \rightarrow \mathbf{R}$ be a function of a single spin configuration. We say that f is increasing when $\sigma(t) \leq \sigma'(t)$ for almost every $t \in [0, \beta]$ implies $f(\sigma) \leq f(\sigma')$. When ν, ν' are two probability measures on Σ such that, for all increasing function f , $\nu(f) \leq \nu'(f)$, we say that ν is stochastically smaller than ν' and we will write $\nu < \nu'$.

Theorem 3.2. *Let $\beta, M < \infty$. There exists $c > 0$ such that, for any $\mathbf{h}, \mathbf{h}', \lambda$ such that $\|\mathbf{h}\|_1, \|\mathbf{h}'\|_1, \lambda < M$, for any $\pi \in \{\emptyset, \text{per}\}$, if $\mathbf{h}' \geq \mathbf{h}$ point-wise a.s. then $\mu_{\mathbf{h}'}^\pi \succ \mu_{\mathbf{h}}^\pi$. Moreover, for any f increasing,*

$$\mu_{\mathbf{h}'}^\pi(f) - \mu_{\mathbf{h}}^\pi(f) \geq c(f(+) - f(-))(\mathbf{h}' - \mathbf{h}) \bullet \mathbf{1}. \quad (3.3)$$

A consequence of monotonicity is the FKG inequality:

Corollary 3.3. *Let $\beta, \lambda < \infty$ and \mathbf{h} with $\|\mathbf{h}\|_\infty < \infty$. Then, for any $\pi \in \{\emptyset, \text{per}\}$, any f, g increasing,*

$$\mu^\pi(fg) \geq \mu^\pi(f)\mu^\pi(g) \quad (3.4)$$

The main idea in the proof of Theorem 3.2 is an explicit coupling of two single site measures with the same parameters but different imaginary time boundary condition.

Lemma 3.4. *Fix $\beta, \lambda > 0$ and $\mathbf{h} \in L^1([0, \beta])$. Consider two independent spins variables $\sigma_1^+ \sim \mu(\cdot | \sigma(\beta) = +)$ and $\sigma_1^- \sim \mu(\cdot | \sigma(\beta) = -)$. Call T^+ (respectively T^-) the last flipping time (0 if none) of σ_1^+ (respectively of σ_1^-), and $T = \max(T^+, T^-)$. Consider the joint distribution Ψ on (σ^+, σ^-) as follows:*

- (1) Let $\sigma_{(T, \beta]}^+ = +$ and $\sigma_{(T, \beta]}^- = -$.
- (2) Take $\sigma_{[0, T)}^+ = \sigma_{[0, T)}^- \sim \mu(\cdot | \sigma(T) = \varepsilon)$ where $\varepsilon = +$ if $T^+ < T^-$, $\varepsilon = -$ otherwise.

Then Ψ is a monotone coupling of $\mu(\cdot | \sigma(\beta) = +)$ and $\mu(\cdot | \sigma(\beta) = -)$. It satisfies

$$\Psi(\sigma^+ = +, \sigma^- = -) = \mu(\sigma = + | \sigma(\beta) = +) \times \mu(\sigma = - | \sigma(\beta) = -). \quad (3.5)$$

Proof of Lemma 3.4. Ψ is monotone by construction ($\sigma^+ \geq \sigma^-$ point-wise a.s.) while (3.5) is a consequence of $\sigma^\pm = \pm \Leftrightarrow T^+ = T^- = 0$. In order to prove that Ψ is a coupling we need to check that it has the correct marginals. Let $t \in (0, \beta)$. Conditioning on $T^+ < t$ is the same as conditioning the Poisson point process on having no point in (t, β) and therefore $\sigma \sim \mu(\cdot | \sigma(\beta) = +, T^+ < t)$, restricted to $(0, t)$, has distribution $\mu_{t, \mathbf{h}, \lambda}(\cdot | \sigma(t) = +)$. Conditioning on $T^+ = t$ is more delicate as this is a zero probability event, still it has a very precise meaning in terms of the Poisson point process, since it requires that there is a point at t and not point in (t, β) . It is well known that the conditional distribution of the points in $(0, t)$ is an independent Poisson point process with the same intensity. Therefore, $\sigma \sim \mu(\cdot | \sigma(\beta) = +, T^+ = t)$, restricted to $(0, t)$, has distribution $\mu_{t, \mathbf{h}, \lambda}(\cdot | \sigma(t) = -)$. The same holds if we replace $+$ by $-$ and vice-versa. Since T^+ and T^- are independent, this proves that Ψ is indeed a coupling of $\mu(\cdot | \sigma(\beta) = +)$ and $\mu(\cdot | \sigma(\beta) = -)$. \square

Proof of Theorem 3.2. We begin with the case without periodic imaginary time boundary conditions, that is $\pi = \emptyset$. It is enough to quantify the influence of a unitary increase of the field, so we consider $\Delta \mathbf{h} : [0, \beta] \rightarrow \mathbf{R}^+$. We remark that

$$\left[\frac{d}{ds} \mu_{\mathbf{h} + s \Delta \mathbf{h}}(f) \right]_{s=0} = \int_0^\beta dt \Delta \mathbf{h}(t) \text{Cov}_{\mu_{\mathbf{h}}}(\sigma(t), f(\sigma)) \quad (3.6)$$

and then

$$\begin{aligned}
\text{Cov}_{\mu_h}(\sigma(t), f(\sigma)) &= \frac{1}{2} \int \mu_h(d\sigma) \mu_h(d\sigma') (\sigma(t) - \sigma'(t)) (f(\sigma) - f(\sigma')) \\
&= 2 \int_{\sigma(t)=+, \sigma'(t)=-} \mu_h(d\sigma) \mu_h(d\sigma') (f(\sigma) - f(\sigma')) \\
&= 2\mu_h(\sigma(t)=+) \mu_h(\sigma(t)=-) [\mu_h(f|\sigma(t)=+) - \mu_h(f|\sigma(t)=-)]
\end{aligned}$$

Conditionally on $\sigma(t)$, the spin on $[0, t]$ and $[t, \beta]$ are independent. So if we take the product of two couplings as in Lemma 3.4 on these two time intervals, we obtain a monotone coupling Ψ of $\mu(\cdot|\sigma(t)=+)$ and $\mu(\cdot|\sigma(t)=-)$ with the property that

$$\begin{aligned}
\Psi(\sigma^+ = +, \sigma^- = -) &= \prod_{\varepsilon=\pm} \mu_{t,h,\lambda}(\sigma = \varepsilon | \sigma(t) = \varepsilon) \mu_{(t,\beta),h,\lambda}(\sigma = \varepsilon | \sigma(t) = \varepsilon) \\
&= \mu(\sigma = + | \sigma(t) = +) \mu(\sigma = - | \sigma(t) = -).
\end{aligned}$$

Furthermore,

$$\begin{aligned}
\mu(f|\sigma(t)=+) - \mu(f|\sigma(t)=-) &= \Psi(f(\sigma^+) - f(\sigma^-)) \\
&\geq \Psi(\sigma^+ = +, \sigma^- = -)(f(+) - f(-))
\end{aligned}$$

so we conclude that

$$\text{Cov}_{\mu_h}(\sigma(t), f(\sigma)) \geq 2\mu_h(\sigma=+) \mu_h(\sigma=-) (f(+) - f(-)) \quad (3.7)$$

which, in combination with (3.6), proves the statement (3.3) when $\pi = \emptyset$. It is not difficult to derive an explicit constant as $\varphi(\sigma=+) = \varphi(\sigma=-) = \exp(-\beta\lambda)/2$, the density of μ with respect to φ being bounded. Now we address the case of periodic imaginary time boundary conditions. We call $\mu_h^{\varepsilon, \varepsilon'} = \mu_h(\cdot | \sigma(0) = \varepsilon, \sigma(\beta) = \varepsilon')$ the measure μ_h conditioned on $\sigma(0) = \varepsilon$ and $\sigma(\beta) = \varepsilon'$. It is clear that

$$\mu_h^{\text{per}} = p\mu_h^{++} + (1-p)\mu_h^{--}. \quad (3.8)$$

where

$$p = p_h = \frac{\mu_h(f^+)}{\mu_h(f^+) + \mu_h(f^-)}.$$

and $f^\varepsilon(\sigma) = \mathbf{1}_{\{\sigma(0)=\sigma(\beta)=\varepsilon\}}$, for $\varepsilon = \pm$. Note that f^+ and $-f^-$ are increasing, and their amplitude is $\varepsilon f^\varepsilon(+) - \varepsilon f^\varepsilon(-) = 1$. When we take derivatives, it is a consequence of (3.3) for free imaginary time boundary condition that

$$\begin{aligned}
\left[\frac{dp_{h+s\Delta h}}{ds} \right]_{s=0} &= \left[\frac{d}{ds} \mu_{h+s\Delta h}(f^+) \right]_{s=0} \times \frac{\mu_h(f^-)}{[\mu_h(f^+) + \mu_h(f^-)]^2} \\
&\quad + \left[\frac{d}{ds} \mu_{h+s\Delta h}(-f^-) \right]_{s=0} \times \frac{\mu_h(f^+)}{[\mu_h(f^+) + \mu_h(f^-)]^2} \\
&\geq c \frac{\int_0^\beta dt \Delta h(t)}{\mu_h(\sigma(0) = \sigma(\beta))}.
\end{aligned} \quad (3.9)$$

Now we take derivatives in (3.8):

$$\begin{aligned}
\left[\frac{d}{ds} \mu_{h+s\Delta h}^{\text{per}}(f) \right]_{s=0} &= \left[\frac{dp_{h+s\Delta h}}{ds} \right]_{s=0} (\mu_h^{++}(f) - \mu_h^{--}(f)) \\
&\quad + p \frac{d}{ds} \mu_{h+s\Delta h}^{++}(f) + (1-p) \frac{d}{ds} \mu_{h+s\Delta h}^{--}(f).
\end{aligned} \quad (3.10)$$

As the imaginary time boundary condition can be realized by adding to \mathbf{h} an additional field $\varepsilon A \mathbf{k}_\delta$, where $\mathbf{k}_\delta = \mathbf{1}_{[0, \delta] \cup [\beta - \delta, \beta]}$, in the limit $A \rightarrow +\infty$ and $\delta \rightarrow 0$ (in that order), the last two terms are non-negative according to (3.3) for $\pi = \emptyset$. So in order to conclude the proof of (3.3) for periodic imaginary time boundary condition it is enough to provide a lower bound on $\mu_{\mathbf{h}}^{++}(f) - \mu_{\mathbf{h}}^{--}(f)$. We know that

$$\mu_{\mathbf{h}}^{\varepsilon\varepsilon}(f) = \lim_{\delta \rightarrow 0} \lim_{A \rightarrow +\infty} \mu_{\mathbf{h} + \varepsilon A \mathbf{k}_\delta}(f).$$

Furthermore, according to (3.3) for $\pi = \emptyset$, for any $\delta > 0$, $A \in \mathbf{R} \mapsto \mu_{\mathbf{h} + A \mathbf{k}_\delta}(f)$ is a increasing function and, on the interval $[-1/\delta, 1/\delta]$ its derivative is not smaller than $c' \delta (f(+) - f(-))$. This proves that

$$\lim_{A \rightarrow +\infty} \mu_{\mathbf{h} + A \mathbf{k}_\delta}(f) - \lim_{A \rightarrow +\infty} \mu_{\mathbf{h} - A \mathbf{k}_\delta}(f) \geq c' (f(+) - f(-)) \quad (3.11)$$

for any $\delta > 0$, where c' does not depend on δ , and consequently $\mu_{\mathbf{h}}^{++}(f) - \mu_{\mathbf{h}}^{--}(f)$ satisfies the same lower bound. Putting (3.11) with (3.9) into (3.10) we obtain (3.3) for periodic imaginary time boundary condition. \square

Proof of Corollary 3.3. Fix $\varepsilon \ll 1$ and assume first that f, g are increasing functions of a single spin, measurable w.r.t. $\mathcal{F}_{T+\varepsilon}$, for some $T \in [0, \beta]$, where \mathcal{F}_t is the σ -algebra generated by $\{\sigma(s)\}_{s \leq t}$. We prove that

$$\text{Cov}(fg | \mathcal{F}_T) \geq -C\varepsilon^2 \|f\|_\infty \|g\|_\infty \quad (3.12)$$

where

$$\text{Cov}(f, g | \mathcal{F}_T) = \mu(fg | \mathcal{F}_T) - \mu(f | \mathcal{F}_T) \mu(g | \mathcal{F}_T).$$

Indeed, call ν the distribution μ conditioned to $\{\sigma(t)\}_{t \leq T}$ and to the event that σ has at most one flip in $[T, T + \varepsilon]$. The distribution ν is completely described by the law of the time of the flip ($+\infty$ if no flip). But f and g are both monotone functions of this random time (both increasing or decreasing), so it follows from the FKG inequality for distributions on the real line (Lemma 16.2 in the lectures notes by Peres [8]) that $\nu(fg) \geq \nu(f) \nu(g)$. But the total variation distance between ν and $\mu(\cdot | \sigma(t), t \leq T)$ is less than $C\varepsilon^2$, which proves (3.12).

Now we take two arbitrary increasing functions f, g of a single spin and choose $T = \beta - \varepsilon$. When we apply the standard formula for conditional covariance together with (3.12) we get that

$$\begin{aligned} \text{Cov}(f, g) &= \mu(\text{Cov}(f, g | \mathcal{F}_T)) + \text{Cov}(\mu(f | \mathcal{F}_T), \mu(g | \mathcal{F}_T)) \\ &\geq -C\varepsilon^2 \|f\|_\infty \|g\|_\infty + \text{Cov}(\mu(f | \mathcal{F}_T), \mu(g | \mathcal{F}_T)) \end{aligned}$$

where $\mu(f | \mathcal{F}_T)$ and $\mu(g | \mathcal{F}_T)$ are increasing functions with infinite norm less than that of $\|f\|_\infty$ and $\|g\|_\infty$, respectively. By applying (3.12) repeatedly with $T = \beta - k\varepsilon$, $k = 2, \dots, \beta/\varepsilon$ we conclude that

$$\text{Cov}(f, g) \geq -C\beta\varepsilon \|f\|_\infty \|g\|_\infty$$

and the claim follows by letting $\varepsilon \rightarrow 0$. \square

4. GLAUBER DYNAMICS FOR THE QUANTUM ISING MODEL

4.1. Definition of the generator and the semi-group. Here we define the Glauber dynamics for finite graphs and establish some preliminary properties.

The dynamics consists in resampling spins locally, according to the field generated by their neighbors. Given the graph G and the parameters $\beta > 0, \lambda \geq 0, \mathbf{h} : V \times [0, \beta] \mapsto \mathbf{R}$ integrable, we call $\mu = \mu_{G;\beta,\mathbf{h},\lambda}$ the Gibbs measure on G with corresponding parameters and

$$\mu_x^\rho = \mu(\cdot | \sigma_y = \rho_y, \forall y \in V \setminus \{x\}), \quad \forall \rho \in \Sigma^V, \forall x \in V.$$

Note that μ_x takes into account both the field \mathbf{h} and the boundary condition on V . According to the DLR equation, μ_x^ρ is the measure obtained by taking $\sigma = \rho$ on $V \setminus \{x\}$ and σ_x according to the Gibbs measure on $\{x\}$ with field $\mathbf{h}_x + \sum_{y \sim x} \rho_y$. We can interpret μ_x as a kernel, since for each $\rho \in \Sigma^V$ and $x \in V$, μ_x^ρ is a probability measure (furthermore, $\rho \mapsto \mu_x^\rho$ is continuous, cf. (3.2)). Because μ_x^ρ is a conditional expectation, it is a contraction in $L^2(\mu)$.

Next we define the generator of the Glauber dynamics by

$$\mathcal{L} = \sum_{x \in V} (\mu_x - I)$$

where I is the identity operator. This is clearly a bounded operator on $L^2(\mu_{G;\beta,\mathbf{h},\lambda})$ and the associated Markov semi-group is

$$\begin{aligned} P_t &= e^{t\mathcal{L}} \\ &= \sum_{n \geq 0} e^{-t|V|} \frac{(t|V|)^n}{n!} \times \left(\frac{1}{|V|} \right)^n \sum_{x_1, \dots, x_n \in V} \mu_{x_1} \cdots \mu_{x_n}. \end{aligned} \quad (4.1)$$

According to its definition, P_t is a bounded operator on $L^2(\mu)$. Equation (4.1) shows as well that, for any $t \geq 0$, P_t is a convex combination of the iterates of μ_x , and is therefore a Markov kernel that contracts L^2 . For any $\rho \in \Sigma^V$, we will write P_t^ρ for the probability measure on Σ^V which, on bounded functions f , acts as $P_t^\rho(f) = (P_t f)(\rho)$. If ρ is distributed according to a probability measure ν on Σ , we will write νP_t for the measure $\int P_t^\rho d\nu(\rho)$.

4.2. Mixing time and spectral gap. In this section, in analogy with the classical situation, we prove some basic results that are useful to control the relaxation of the dynamics to the equilibrium Gibbs measure.

Proposition 4.1. *Let some $G = (V, E)$ finite, $\beta \geq 0, \mathbf{h} : V \times [0, \beta] \rightarrow \mathbf{R}$ integrable and $\lambda \geq 0$. Define*

$$T_{\text{mix}} = \inf \{ t \geq 0 : \forall \rho, \eta \in \Sigma^V, \|P_t^\rho - P_t^\eta\|_{\text{TV}} \leq e^{-1} \}.$$

Then $T_{\text{mix}} < \infty$ and, for any $t \geq 0$,

$$\sup_{\rho \in \Sigma^V} \|P_t^\rho - \mu\|_{\text{TV}} \leq \sup_{\rho, \eta \in \Sigma^V} \|P_t^\rho - P_t^\eta\|_{\text{TV}} \leq e^{-\lfloor t/T_{\text{mix}} \rfloor}.$$

Proof. The first inequality is a consequence of μ being invariant by P_t (in other words, μ is a convex combination of the P_t^η). The second inequality is classical consequence of

$$\bar{d}(t) = \sup_{\rho, \eta \in \Sigma^V} \|P_t^\rho - P_t^\eta\|_{\text{TV}}$$

being sub-multiplicative, cf. [5]. It remains to prove that $T_{\text{mix}} < \infty$, or equivalently that $\bar{d}(t) < 1$ for some $t > 0$. This follows from (4.1) once we remark that any $\mu_{x_1} \cdots \mu_{x_n}$ with $\{x_1, \dots, x_n\} = V$ gives a probability at least $c^{-|V|}$ to the uniform plus state, uniformly in the starting state ρ . \square

Proposition 4.2. *Let some $G = (V, E)$ finite, $\beta \geq 0$, $\mathbf{h} : V \times [0, \beta] \rightarrow \mathbf{R}$ integrable and $\lambda \geq 0$. Define*

$$\text{gap}(\mathcal{L}) = \inf_{f \in L^2(\mu) : \text{Var}(f) > 0} \frac{\text{Cov}(f, -\mathcal{L}f)}{\text{Var}(f^2)}$$

where Cov and Var refer, respectively, to the covariance and the variance under μ . Then

(1) *There exists $c < \infty$ depending on β, λ and $\|\mathbf{h}\|_\infty$ such that*

$$\text{gap}(\mathcal{L}) \geq c^{-|V|}.$$

(2) *For any $f \in L^2(\mu)$,*

$$\text{Var}(P_t f) \leq e^{-2t \text{gap}(\mathcal{L})} \text{Var}(f). \quad (4.2)$$

Proof. The proof of the second point is standard. For the first one we refer the reader to the proof of Theorem 6.4 in the Saint Flour course [6]. \square

4.3. Monotonicity. Now we address the question of the monotonicity of the dynamics. An immediate consequence of (4.1) together with the monotonicity of the single site measure (Theorem 3.2) is the following fact:

Proposition 4.3. *Take $\rho, \eta \in \Sigma^V$ such that $\rho \leq \eta$, and $\mathbf{h} \leq \tilde{\mathbf{h}}$. Denote by \tilde{P}_t the semi-group corresponding to field $\tilde{\mathbf{h}}$. Then,*

$$P_t^\rho \leq_{\text{stoch.}} \tilde{P}_t^\eta.$$

According to the convergence towards the equilibrium measure, it follows that

Corollary 4.4. $\mu_{\mathbf{h}}$ *increases stochastically with the field \mathbf{h} .*

Remark 4.5. The same argument as above could be used to establish the existence of a grand coupling, but for this we would need to know the existence of a grand coupling for the family of single spin measures, given an arbitrary family of external fields.

4.4. Censoring. For any $A \subset V$, we let

$$\mathcal{L}_A = \sum_{x \in A} (\mu_x - I).$$

Now we consider a function $A : \mathbf{R}^+ \rightarrow \mathcal{P}(V)$ with finitely many discontinuities at $t_0 = 0 < t_1 < \dots < t_n$. We define the censored dynamics according to A by the kernel

$$P_{A;t} = e^{(t_1-t_0)\mathcal{L}_{A_0}} e^{(t_2-t_1)\mathcal{L}_{A_1}} \dots e^{(t-t_k)\mathcal{L}_{A_k}} \quad (4.3)$$

where k is the largest integer in $\{0, \dots, n\}$ such that $t \geq t_k$, and $A_i = A(t_i^+)$. Of course, when $A(t) = V$ for any $t \geq 0$ we get $P_{A;t} = P_t$, the uncensored dynamics. The theory of censoring due to Peres and Winkler [8] also applies here. Remarkably their result on total variation extends also to variance and entropy.

Proposition 4.6. *Consider $A, B : \mathbf{R}^+ \rightarrow \mathcal{P}(V)$ as above. Assume that, for any $t \geq 0$, $A(t) \subset B(t)$. Assume that ν is absolutely continuous with respect to μ with $d\nu/d\mu \in L^2(d\mu)$ and increasing. Then, for any $t \geq 0$, both $\nu P_{A;t}$ and $\nu P_{B;t}$ are absolutely continuous with respect*

to μ , their Radon-Nikodym derivative is increasing and $\nu P_{B;t} \prec \nu P_{A;t}$. Moreover the following inequalities hold:

$$\text{Var} \left(\frac{d(\nu P_{B;t})}{d\mu} \right) \leq \text{Var} \left(\frac{d(\nu P_{A;t})}{d\mu} \right) \quad (4.4)$$

$$\text{Ent} \left(\frac{d(\nu P_{B;t})}{d\mu} \right) \leq \text{Ent} \left(\frac{d(\nu P_{A;t})}{d\mu} \right) \quad (4.5)$$

$$\|\nu P_{B;t} - \mu\|_{\text{TV}} \leq \|\nu P_{A;t} - \mu\|_{\text{TV}}. \quad (4.6)$$

Remark 4.7. A special case satisfying the assumptions of the proposition is when ν is concentrated on the identical equal to $+$ configuration. In that case we will write $\nu P_{A;t} = P_{A;t}^+$ and $\nu P_{B;t} = P_{B;t}^+$.

Following [8] we begin the proof with two lemmas.

Lemma 4.8. *Consider ν some measure on Σ^V , and assume that ν is absolutely continuous with respect to μ with $\frac{d\nu}{d\mu}$ being an increasing function. Then $\nu\mu_x$ is absolutely continuous with respect to μ and its Radon-Nikodym derivative is increasing as well.*

Proof. Let $\sigma \leq \tau$ and assume that $d\nu/d\mu$ is increasing. We denote by σ^* the spin configuration on $V \setminus \{x\}$ equal to σ on $V \setminus \{x\}$, and by σ_x^ξ the spin configuration on V equal to ξ at x and to σ on $V \setminus \{x\}$. Then

$$\begin{aligned} \frac{d(\nu\mu_x)}{d\mu}(\sigma) &= \frac{d\nu}{d\mu}(\sigma^*) = \int \mu(d\xi|\sigma^*) \frac{d\nu}{d\mu}(\sigma_x^\xi) \\ &\leq \int \mu(d\xi|\sigma^*) \frac{d\nu}{d\mu}(\tau_x^\xi) \\ &\leq \int \mu(d\xi|\tau^*) \frac{d\nu}{d\mu}(\tau_x^\xi) = \frac{d(\nu\mu_x)}{d\mu}(\tau) \end{aligned}$$

where we used, in the first line, the fact that the density does not depend on σ_x since that spin is resampled, and at the second line the assumption that $d\nu/d\mu$ is increasing, and finally the fact that the single spin marginal increases with the external field (Theorem 3.2). \square

Lemma 4.9. *Consider ν some measure on Σ^V , and assume that ν is absolutely continuous with respect to μ with $\frac{d\nu}{d\mu}$ being an increasing function. Then $\nu\mu_x \prec \nu$.*

Proof. Contrary to [8] the set of single spins configurations is not totally ordered. For this reason we use an alternative proof based on the FKG inequality for single spin measures. Let f increasing. We start with

$$\nu(f) - (\nu\mu_x)(f) = \nu(f - \mu_x(f))$$

and condition on σ^* , the spin configuration outside x . We have

$$\begin{aligned} \nu(f - \mu_x(f) | \sigma^*) &= \mu_x^{\sigma^*} \left(\left[\frac{\frac{d\nu}{d\mu}}{\mu_x^{\sigma^*} \left(\frac{d\nu}{d\mu} \right)} - 1 \right] f \right) \\ &\geq \mu_x^{\sigma^*} \left(\frac{\frac{d\nu}{d\mu}}{\mu_x^{\sigma^*} \left(\frac{d\nu}{d\mu} \right)} - 1 \right) \mu_x^{\sigma^*}(f) \\ &= 0 \end{aligned}$$

where in the second line we use the FKG inequality for a single spin (Corollary 3.3). \square

Proof. (Proposition 4.6). Lemmas 4.8 and 4.9 together with formula (4.1) imply that $\nu P_{A;t}$ and $\nu P_{B;t}$ are absolutely continuous with respect to μ , that their Radon-Nikodym derivative is increasing and also that $\nu P_{B;t} \prec \nu P_{A;t}$. Now we prove the inequalities. For the variance, we remark that

$$\begin{aligned} \text{Var} \left(\frac{d(\nu P_{B;t})}{d\mu} \right) &= \mu \left(\left(\frac{d(\nu P_{B;t})}{d\mu} \right)^2 \right) - 1 = \nu P_{B;t} \left(\frac{d(\nu P_{B;t})}{d\mu} \right) - 1 \\ &\leq \nu P_{A;t} \left(\frac{d(\nu P_{B;t})}{d\mu} \right) - 1 = \text{Cov} \left(\frac{d(\nu P_{A;t})}{d\mu}, \frac{d(\nu P_{B;t})}{d\mu} \right) \\ &\leq \text{Var} \left(\frac{d(\nu P_{A;t})}{d\mu} \right)^{1/2} \text{Var} \left(\frac{d(\nu P_{B;t})}{d\mu} \right)^{1/2} \end{aligned}$$

which proves (4.4). For the entropy we recall that

$$\text{Ent}(f) = \sup \{ \mu(fg) : g \text{ with } \mu(e^g) = 1 \}$$

therefore

$$\begin{aligned} \text{Ent} \left(\frac{d(\nu P_{B;t})}{d\mu} \right) &= (\nu P_{B;t}) \left(\log \frac{d(\nu P_{B;t})}{d\mu} \right) \\ &\leq (\nu P_{A;t}) \left(\log \frac{d(\nu P_{B;t})}{d\mu} \right) = \mu \left(\frac{d(\nu P_{A;t})}{d\mu} \log \frac{d(\nu P_{B;t})}{d\mu} \right) \\ &\leq \text{Ent} \left(\frac{d(\nu P_{A;t})}{d\mu} \right). \end{aligned}$$

Finally we recall for completeness the proof of (4.6):

$$\begin{aligned} \|\nu P_{B;t} - \mu\|_{\text{TV}} &= \nu P_{B;t} \left(\frac{d(\nu P_{B;t})}{d\mu} \geq 1 \right) - \mu \left(\frac{d(\nu P_{B;t})}{d\mu} \geq 1 \right) \\ &\leq \nu P_{A;t} \left(\frac{d(\nu P_{B;t})}{d\mu} \geq 1 \right) - \mu \left(\frac{d(\nu P_{B;t})}{d\mu} \geq 1 \right) \\ &\leq \|\nu P_{A;t} - \mu\|_{\text{TV}}. \end{aligned}$$

□

5. ISING MODEL ON REGULAR TREES

5.1. Notation and main results. In the following we specialize to the case where the underlying graph is \mathbb{T}_l^b , the rooted regular tree with $b \geq 2$ children at each node except the leaves, and depth $l \geq 0$. We always denote by r the root of the tree. When $z \in \mathbb{T}_l^b$ we denote by $|z|$ the depth of z , that is the graph distance to the root. We say that z is a leaf if $|z| = l$. Given $\tau \in \Sigma^{\mathbb{T}_\infty^b}$ and A is a subset of \mathbb{T}_∞^b we define μ_A^τ as the Gibbs measure on Σ^A with boundary condition τ acting as an additional local field at $z \in A$ given by $\sum_{y \notin A, y \sim z} \tau(y)$, where $y \sim z$ means that y, z are neighbors. When τ is identically equal to plus/minus we will simply write μ_A^\pm . When $A = \mathbb{T}_l^b$, we will write μ_l^τ .

Definition 5.1. We say that the parameters $\beta, \lambda \geq 0$, $\mathbf{h} \in L^1([0, \beta])$ are inside the uniqueness region if the boundary condition on the leaves of \mathbb{T}_l^b does not change the marginal distribution at the root in the limit $l \rightarrow +\infty$. Because of stochastic domination, this is equivalent to

$$\lim_{l \rightarrow +\infty} \left\| \mu_l^+(\sigma_r \in \cdot) - \mu_l^-(\sigma_r \in \cdot) \right\|_{\text{TV}} = 0. \quad (5.1)$$

Next, following [7] we define the exponents γ and κ as follows:

Definition 5.2. Given $\beta, \lambda \geq 0$, $\mathbf{h} \in L^1([0, \beta])$ we let

$$\gamma = \sup_l \max \left\| \mu_A^{\boldsymbol{\eta}}(\boldsymbol{\sigma}_z \in \cdot) - \mu_A^{\bar{\boldsymbol{\eta}}}(\boldsymbol{\sigma}_z \in \cdot) \right\|_{\text{TV}} \quad (5.2)$$

where the maximum is taken over all subsets $A \subset \mathbb{T}_l^b$, all vertices y on the external boundary of A , all boundary configurations $\boldsymbol{\eta}, \bar{\boldsymbol{\eta}} \in \Sigma^{\mathbb{T}_\infty^b}$ that differ only at y and all neighbors $z \in A$ of y .

Given a boundary condition $\boldsymbol{\tau} \in \Sigma^{\mathbb{T}_\infty^b}$, we let $\kappa(\boldsymbol{\tau})$ be the infimum of $\kappa \geq 0$ such that there exists $l_0 \geq 0$ such that, for any regular subtree $T \subset \mathbb{T}_\infty^b$ with root x and uniform depth, and for any $z \in T$ with $|z - x| \geq l_0$,

$$\mu_T^{\boldsymbol{\tau}}(\boldsymbol{\sigma}_z \bullet 1 | \boldsymbol{\sigma}_x = +) - \mu_T^{\boldsymbol{\tau}}(\boldsymbol{\sigma}_z \bullet 1 | \boldsymbol{\sigma}_x = -) \leq \kappa^{|z|}. \quad (5.3)$$

From Proposition 3.1 we know already that $\gamma < 1$. It follows from a recursive coupling argument that $\kappa \leq \gamma$. In complete analogy with the classical case, we prove the following results.

Theorem 5.3. (*Decay of correlations*) Let $\beta, \lambda \geq 0$, $\mathbf{h} \in L^1([0, \beta])$.

- (1) $\kappa(+) \leq 1/b$.
- (2) If $\beta, \lambda, \mathbf{h}$ are in the uniqueness region, then for arbitrary $\boldsymbol{\tau} \in \Sigma^{\mathbb{T}_\infty^b}$, $\kappa(\boldsymbol{\tau}) \leq 1/b$.

Theorem 5.4. (*Fast mixing*) Let $\beta, \lambda \geq 0$, $\mathbf{h} \in L^1([0, \beta])$. Fix $\boldsymbol{\tau} \in \Sigma^{\mathbb{T}_\infty^b}$ and assume that $\kappa(\boldsymbol{\tau})$ is such that $\kappa(\boldsymbol{\tau})\gamma b < 1$. Then the following holds.

- (1) The spectral gap of the Glauber dynamics on \mathbb{T}_l^b with boundary condition $\boldsymbol{\tau}$ is greater than a positive constant which does not depend on $l \geq 0$.
- (2) The mixing time corresponding to the above dynamics is at most Cl where $C < \infty$ does not depend on $l \geq 0$.

5.2. Conditional spin distributions. Working with trees leads to major simplifications in the structure of the Gibbs measure. In particular, given any $z \in \mathbb{T}$, the restriction of a Gibbs measure μ onto the subtrees of z , given $\boldsymbol{\sigma}_z$, is a product measure. As a consequence, μ is fully characterized by

- (1) the marginal distribution $\mu(\boldsymbol{\sigma}_r \in \cdot)$ of the spin at the root r ,
- (2) and the conditional marginal distributions $\mu(\boldsymbol{\sigma}_z \in \cdot | \boldsymbol{\sigma}_{z^-})$, $z \in \mathbb{T} \setminus \{r\}$, where z^- denotes the ancestor of z .

Remark 5.5. The marginal distribution $\mu(\boldsymbol{\sigma}_r \in \cdot)$ can be viewed as a conditional spin distribution if we add a ghost ancestor 0 to the root r with constant spin $\boldsymbol{\sigma}_0 = 0$.

For latter purposes we need that (conditional) measures form a vector space. So we introduce the set \mathcal{M} of finite signed measures on $(\Sigma, \mathcal{B}(\Sigma))$, where $\mathcal{B}(\Sigma)$ is the Borel σ -algebra associated to the Skorohod topology on Σ . Signed measures have a Hahn-Jordan decomposition into their positive and negative parts. This means that, for any $\mu \in \mathcal{M}$, there are two disjoint sets P and N (unique up to μ -negligible sets) such that μ gives non-negative weight to every Borel subset of P , and non-positive weight to every Borel subset of N . Let $\mu^+ = \mu(\cdot \cap P)$ and $\mu^- = -\mu(\cdot \cap N)$. Then $\mu = \mu^+ - \mu^-$. We also consider $|\mu| = \mu^+ + \mu^-$, a positive measure, and recall that

$$\|\mu\|_{\text{TV}} = \frac{|\mu|(\Sigma)}{2} = \frac{1}{2} \sup_{f: |f| \leq 1} \mu(f) = \frac{1}{2} \sup_{A \subset \Sigma} (\mu(A) - \mu(A^c)) \quad (5.4)$$

defines a norm on \mathcal{M} , that turns \mathcal{M} into a Banach space as Σ endowed with the Skorohod topology is a Polish space.

For convenience, we call \mathcal{M}_0 the subset of \mathcal{M} made of all finite signed measures μ with $\mu(\Sigma) = 0$. Both \mathcal{M} and \mathcal{M}_0 are vector spaces. We also call \mathcal{M}_+ the set of finite positive measures and $\mathcal{M}_{+,1}$ the set of probability measures on Σ .

Finally, we call \mathcal{X} the set of functions from Σ to \mathcal{M} , \mathcal{X}_0 the set of functions from Σ to \mathcal{M}_0 , and similarly $\mathcal{X}_{+,1}$ for the set of functions from Σ to $\mathcal{M}_{+,1}$. In particular, any marginal (resp. conditional marginal) distribution belongs to $\mathcal{M}_{+,1}$ (resp. $\mathcal{X}_{+,1}$), and any difference of marginal (resp. conditional marginal) distributions belongs to \mathcal{M}_0 (resp. \mathcal{X}_0). For notation consistence, for any $\rho \in \mathcal{X}$ and any $\sigma \in \Sigma$, we denote by $\rho^\sigma(\cdot)$ the corresponding signed measure on Σ . We will consider later on the problem of defining a norm on \mathcal{X} (see (5.14) and (5.15)).

5.3. The resampling operator and the cavity equation. Given $\eta \in L^1([0, \beta])$, $\rho \in \mathcal{M}$ and $\rho_1, \dots, \rho_b \in \mathcal{X}$, we define $R_{\rho_1, \dots, \rho_b}^\eta(\rho) \in \mathcal{M}$ by

$$R_{\rho_1, \dots, \rho_b}^\eta(\rho) = \int \rho(d\sigma_0) \rho_1^{\sigma_0}(d\sigma_1) \cdots \rho_b^{\sigma_0}(d\sigma_b) \mu_{\beta, \mathbf{h} + \eta + \sigma_1 + \dots + \sigma_b, \lambda}. \quad (5.5)$$

We give an interpretation to $R_{\rho_1, \dots, \rho_b}^\eta(\rho)$ in an important special case.

- (1) Consider some point of the tree \mathbb{T} with $b + 1$ neighbors. Conventionally we denote its spin by σ_0 . We call η the spin of its ancestor and $\sigma_1, \dots, \sigma_b$ the spins of its children.
- (2) Sample σ_0 from ρ , the conditional distribution of σ_0 given the neighbor spin η .
- (3) Sample each σ_i independently, according to $\rho_i^{\sigma_0}$, the conditional distribution of σ_i given the parent spin σ_0 .
- (4) Sample again σ_0 according to the single site distribution with field $\mathbf{h} + \eta + \sigma_1 + \dots + \sigma_b$.

If ρ and the ρ_i correspond to the conditional marginal distributions of the Gibbs measure with field \mathbf{h} , then ρ is stable under the resampling operator according to the DLR equation. In other words, $\rho = R_{\rho_1, \dots, \rho_b}^\eta(\rho)$ when ρ is the conditional distribution of σ_0 given that the spin above 0 is η , and the $\rho_i^{\sigma_0}$ are the conditional distributions of σ_i given σ_0 . More generally we have:

Definition 5.6. Given $\rho_1, \dots, \rho_b \in \mathcal{X}$, $\eta \in L^1([0, \beta])$, we say that $\rho \in \mathcal{M}$ satisfies the *cavity equation* with parameters ρ_1, \dots, ρ_b and η if

$$\rho = R_{\rho_1, \dots, \rho_b}^\eta(\rho) \quad (5.6)$$

5.3.1. The resampling operator is a contraction. From the definition (5.5) it is clear that the applications $\rho \in \mathcal{M} \mapsto R_{\rho_1, \dots, \rho_b}^\eta(\rho) \in \mathcal{M}$ and $\rho_i \in \mathcal{X} \mapsto R_{\rho_1, \dots, \rho_b}^\eta(\rho) \in \mathcal{M}$ are linear. Now we consider their operator norm.

Proposition 5.7. *Let γ be the constant in Proposition 3.1 corresponding to the value of M given by $M = \max(\lambda, \|\mathbf{h}\|_1 + \beta(b + 1))$. Then,*

- (1) *For any $\rho_1, \dots, \rho_b \in \mathcal{X}_{1,+}$, any $\eta \in \Sigma$, the application $R_{\rho_1, \dots, \rho_b}^\eta$ is a contraction on \mathcal{M} , and a γ -contraction (i.e. its operator norm is at most γ) when restricted to \mathcal{M}_0 .*
- (2) *For any $\rho_1, \dots, \rho_b \in \mathcal{X}_{1,+}$, any $\eta \in \Sigma$, the application $I - R_{\rho_1, \dots, \rho_b}^\eta$ is invertible on \mathcal{M}_0 and its inverse*

$$(I - R_{\rho_1, \dots, \rho_b}^\eta)^{-1} = \sum_{k \geq 0} (R_{\rho_1, \dots, \rho_b}^\eta)^k \quad (5.7)$$

has operator norm not larger than $(1 - \gamma)^{-1}$.

- (3) *For any $\rho \in \mathcal{M}$, $\rho_1, \dots, \rho_b \in \mathcal{X}$,*

$$\|R_{\rho_1, \dots, \rho_b}^\eta(\rho)\|_{\text{TV}} \leq \int |\rho|(d\sigma_0) \prod_{i=1}^b \|\rho_i^{\sigma_0}\|_{\text{TV}}. \quad (5.8)$$

Proof. (1) We write $\rho = \rho^+ - \rho^-$ the Hahn-Jordan decomposition of ρ into positive, mutually singular measures. We have

$$\|R_{\rho_1, \dots, \rho_b}^\eta(\rho)\|_{\text{TV}} \leq \|R_{\rho_1, \dots, \rho_b}^\eta(\rho^+)\|_{\text{TV}} + \|R_{\rho_1, \dots, \rho_b}^\eta(\rho^-)\|_{\text{TV}}$$

But $R_{\rho_1, \dots, \rho_b}^\eta(\rho^+)$ is obviously a positive measure, and its mass is

$$\begin{aligned} R_{\rho_1, \dots, \rho_b}^\eta(\rho^+)(\Sigma) &= \int \rho^+(d\sigma_0) \rho_1^{\sigma_0}(d\sigma_1) \cdots \rho_b^{\sigma_0}(d\sigma_b) \\ &= \rho^+(\Sigma) \end{aligned}$$

therefore $\|R_{\rho_1, \dots, \rho_b}^\eta(\rho)\|_{\text{TV}} \leq \|\rho\|_{\text{TV}}$. Now we assume that $\rho \in \mathcal{M}_0$, that is, $\rho^+(\Sigma) = \rho^-(\Sigma)$. The same calculation as above shows that $R_{\rho_1, \dots, \rho_b}^\eta(\rho) \in \mathcal{M}_0$. Now, given $\sigma_0, \eta \in \Sigma$ we consider the probability measure

$$\varphi^{\eta, \sigma_0} = \int \rho_1^{\sigma_0}(d\sigma_1) \cdots \rho_b^{\sigma_0}(d\sigma_b) \mu_{\beta, \mathbf{h} + \eta + \sigma_1 + \cdots + \sigma_b, \lambda}.$$

It is immediate from (3.1) that

$$|\varphi^{\eta, \sigma_0}(A) - \varphi^{\eta, \sigma'_0}(A)| \leq \gamma, \quad \forall \eta, \sigma_0, \sigma'_0 \in \Sigma, \forall A \in \mathcal{B}(\Sigma).$$

Consequently,

$$\begin{aligned} & R_{\rho_1, \dots, \rho_b}^\eta(\rho)(A) - R_{\rho_1, \dots, \rho_b}^\eta(\rho)(A^c) \\ &= \int \rho^+(d\sigma_0) [\varphi^{\eta, \sigma_0}(A) - \varphi^{\eta, \sigma_0}(A^c)] - \int \rho^-(d\sigma_0) [\varphi^{\eta, \sigma_0}(A) - \varphi^{\eta, \sigma_0}(A^c)] \\ &= 2 \int \frac{\rho^+(d\sigma_0) \rho^-(d\sigma'_0)}{\rho^+(\Sigma)} [\varphi^{\eta, \sigma_0}(A) - \varphi^{\eta, \sigma'_0}(A)] \\ &\leq 2\gamma \|\rho\|_{\text{TV}} \end{aligned}$$

since $\rho^+(\Sigma) = \rho^-(\Sigma) = \|\rho\|_{\text{TV}}$. This shows that $R_{\rho_1, \dots, \rho_b}^\eta$ is a γ -contraction on \mathcal{M}_0 .

- (2) It is an immediate consequence of the first point that $\rho \in \mathcal{M}_0 \mapsto \rho - R_{\rho_1, \dots, \rho_b}^\eta(\rho) \in \mathcal{M}_0$ is invertible, with the given inverse. The computation of the operator norm of the inverse is immediate.
- (3) This follows at once from the definition of total variation distance :

$$\|\mu\|_{\text{TV}} = \sup_{f: |f| \leq 1} \mu(f)/2.$$

□

5.3.2. Monotonicity of the resampling operator. In this paragraph we examine the monotonicity properties of the resampling operator. We recall that, when $\mu, \nu \in \mathcal{M}_{1,+}$, we say that μ is stochastically larger than ν when, for all $f : \Sigma \rightarrow \mathbf{R}$ increasing, $\mu(f) \geq \nu(f)$. We generalize this notion by saying that $\rho \in \mathcal{M}_0$ is *stochastically positive* when $\rho(f) \geq 0$, for all f increasing, which we write $\rho \succ 0$. Note that this has nothing to do with “ ρ is a positive measure”, which itself is the same as $\rho \in \mathcal{M}_+$.

Remark 5.8. Let μ, ν two probability measures and assume that μ is stochastically larger than ν . Let $\rho = \mu - \nu$ and call ρ^\pm the positive and negative parts of μ . We claim that a coupling for the positive and negative parts of ρ yields easily a monotone coupling of μ and ν , which is optimal in the sense that it realizes total variation distance. In other words, there always exists an optimal

and monotone coupling between two ordered probability measures. An explicit construction of this coupling is given below. By definition of the total variation distance,

$$\rho^+(\Sigma) = \rho^-(\Sigma) = \|\mu - \nu\|_{\text{TV}}.$$

Assume that $\mu \neq \nu$ and consider φ a monotone coupling of the stochastically ordered probability measures $\|\mu - \nu\|_{\text{TV}}^{-1} \rho^\pm$. Note that $(\mu - \rho^+) = (\nu - \rho^-)$ is a positive measure with mass $1 - \|\mu - \nu\|_{\text{TV}}$. We call ψ the law of (σ, σ) when $\sigma \sim (\mu - \rho^+)/ (1 - \|\mu - \nu\|_{\text{TV}})$. Then,

$$\|\mu - \nu\|_{\text{TV}} \varphi + (1 - \|\mu - \nu\|_{\text{TV}}) \psi$$

is a monotone and optimal coupling for (μ, ν) .

We begin by observing that $R_{\rho_1, \dots, \rho_b}^\eta(\rho)$ is increasing in all parameters $\rho \in \mathcal{M}_{+,1}$, $\rho_1, \dots, \rho_b \in \mathcal{X}_{+,1}$ and η such that for all $i \in \{1, \dots, b\}$, $\rho_i^{\sigma_0}$ increases stochastically with the boundary condition $\sigma_0 \in \Sigma$. More precisely, $R_{\bar{\rho}_1, \dots, \bar{\rho}_b}^{\bar{\eta}}(\rho) \succ R_{\rho_1, \dots, \rho_b}^\eta(\rho)$ if $\rho \in \mathcal{M}_{+,1}$, $\rho_1, \dots, \rho_b, \bar{\rho}_1, \dots, \bar{\rho}_b \in \mathcal{X}_{+,1}$ and $\eta, \bar{\eta} \in L^1([0, \beta])$ satisfy $\bar{\eta} \geq \eta$ and $\bar{\rho}_i^{\sigma_0} \succ \rho_i^{\sigma_0}$, $i = 1 \dots b$, for any $\sigma_0 \in \Sigma$. Also $R_{\rho_1, \dots, \rho_b}^\eta(\bar{\rho}) \succ R_{\rho_1, \dots, \rho_b}^\eta(\rho)$ if $\rho, \bar{\rho} \in \mathcal{M}_{+,1}$, $\rho_1, \dots, \rho_b \in \mathcal{X}_{+,1}$ and $\eta \in L^1([0, \beta])$ are such that $\bar{\rho} \succ \rho$ and $\rho_i^{\sigma_0}$ increases stochastically with $\sigma_0 \in \Sigma$, $i = 1 \dots b$. Both statements follow immediately from equation (5.5) and Theorem 3.2.

It appends that one can compare the stochastic increase in the parameter ρ_1 with the norm of the difference:

Proposition 5.9. *Let $\eta \in \Sigma$. Let $c > 0$ and Γ be the constants in Proposition 3.1 and Theorem 3.2 corresponding to β and $M = \max(\|\mathbf{h}\|_1 + \beta(b+1), \lambda)$. Let $f : \Sigma \rightarrow \mathbf{R}$ be increasing. Assume that $\rho \in \mathcal{M}_{+,1}$, $\rho_1 \in \mathcal{X}_0$ with $\rho_1^{\sigma_0} \succ 0$ for all $\sigma_0 \in \Sigma$ and let $\rho_2, \dots, \rho_b \in \mathcal{X}_{+,1}$. Then*

(1)

$$\|R_{\rho_1, \dots, \rho_b}^\eta(\rho)\|_{\text{TV}} \leq \Gamma \int \rho(d\sigma_0) \rho_1^{\sigma_0}(\sigma \bullet \mathbf{1}).$$

(2)

$$R_{\rho_1, \dots, \rho_b}^\eta(\rho)(f) \geq c(f(+) - f(-)) \int \rho(d\sigma_0) \rho_1^{\sigma_0}(\sigma \bullet \mathbf{1}).$$

Proof. (1) The assumption that $\rho_1^{\sigma_0}$ is stochastically positive means exactly that its positive part is stochastically larger than its negative part. We consider therefore Ψ^{σ_0} a monotone coupling of these two parts:

$$\begin{aligned} & \|R_{\rho_1, \dots, \rho_b}^\eta(\rho)\|_{\text{TV}} \\ &= \frac{1}{2} \sup_{f: |f| \leq 1} \int \rho(d\sigma_0) d\Psi^{\sigma_0}(\sigma_1^+, \sigma_1^-) \rho_2^{\sigma_0}(d\sigma_2) \cdots \rho_b^{\sigma_0}(d\sigma_b) \left[\mu_{\mathbf{h} + \eta + \sigma_1^+ + \dots + \sigma_b}(f) - \mu_{\mathbf{h} + \eta + \sigma_1^- + \dots + \sigma_b}(f) \right] \\ & \leq \Gamma \int \rho(d\sigma_0) d\Psi^{\sigma_0}(\sigma_1^+, \sigma_1^-) \|\sigma_1^+ - \sigma_1^-\|_1 \\ &= \Gamma \int \rho(d\sigma_0) d\Psi^{\sigma_0}(\sigma_1^+, \sigma_1^-) (\sigma_1^+ - \sigma_1^-) \bullet \mathbf{1} \\ &= \Gamma \int \rho(d\sigma_0) \rho_1^{\sigma_0}(\sigma \bullet \mathbf{1}) \end{aligned}$$

(2) Let $F(\sigma_1) = \int \rho_2^{\sigma_0}(d\sigma_2) \cdots \rho_b^{\sigma_0}(d\sigma_b) \mu_{\mathbf{h} + \eta + \sigma_1 + \dots + \sigma_b}(f)$, then $\sigma_1 \mapsto F(\sigma_1)$ is non-decreasing and moreover, for every $\sigma_1^- \leq \sigma_1^+$,

$$F(\sigma_1^+) - F(\sigma_1^-) \geq c(f(+) - f(-))(\sigma_1^+ - \sigma_1^-) \bullet \mathbf{1}$$

according to (3.3) in Theorem 3.2. When we introduce the same coupling of the positive and negative parts of $\rho_1^{\sigma_0}$ as in the former paragraph, we obtain the sought lower bound. \square

5.3.3. The cavity equation has a unique solution.

Theorem 5.10. *Let $\rho_1, \dots, \rho_b \in \mathcal{X}_{+,1}$ and $\boldsymbol{\eta} \in L^1([0, \beta])$. The cavity equation (5.6) has a unique solution in $\mathcal{M}_{+,1}$, that we call $\Phi_{\rho_1, \dots, \rho_b}^{\boldsymbol{\eta}}$. Furthermore, this solution satisfies:*

$$\Phi_{\bar{\rho}_1, \rho_2, \dots, \rho_b}^{\boldsymbol{\eta}} - \Phi_{\rho_1, \dots, \rho_b}^{\boldsymbol{\eta}} = (I - R_{\bar{\rho}_1, \rho_2, \dots, \rho_b}^{\boldsymbol{\eta}})^{-1} R_{\bar{\rho}_1 - \rho_1, \rho_2, \dots, \rho_b}^{\boldsymbol{\eta}} (\Phi_{\rho_1, \dots, \rho_b}^{\boldsymbol{\eta}}), \quad \forall \bar{\rho}_1 \in \mathcal{X}_{+,1}. \quad (5.9)$$

Moreover, the solution to the cavity equation $\Phi_{\rho_1, \dots, \rho_b}^{\boldsymbol{\eta}}$ increases stochastically with its parameters $\rho_1, \dots, \rho_b \in \mathcal{X}_{+,1}$ increasing with the boundary condition, and $\boldsymbol{\eta} \in L^1([0, \beta])$. More precisely, $\Phi_{\bar{\rho}_1, \dots, \bar{\rho}_b}^{\boldsymbol{\eta}} \succ \Phi_{\rho_1, \dots, \rho_b}^{\boldsymbol{\eta}}$ if $\rho_1, \dots, \rho_b, \bar{\rho}_1, \dots, \bar{\rho}_b \in \mathcal{X}_{+,1}$ and $\boldsymbol{\eta}, \bar{\boldsymbol{\eta}} \in L^1([0, \beta])$ satisfy $\bar{\boldsymbol{\eta}} \geq \boldsymbol{\eta}$ and $\bar{\rho}_i^{\sigma_0} \succ \rho_i^{\sigma_0}$, $i = 1 \dots b$, for any $\sigma_0 \in \Sigma$, with both $\bar{\rho}_i^{\sigma_0}$ and $\rho_i^{\sigma_0}$ being stochastically increasing in $\sigma_0 \in \Sigma$.

Proof. We consider first the case when $\rho_i \in \mathcal{X}_{+,1}$ are constant, i.e. ρ_i^{σ} does not depend on $\sigma \in \Sigma$, for all $i = 1 \dots b$. Then, $R_{\rho_1, \dots, \rho_b}^{\boldsymbol{\eta}}(\rho)$ depends uniquely on the mass of $\rho \in \mathcal{M}$. Consequently the unique solution to the cavity equation in $\mathcal{M}_{+,1}$, for this parameters, is $R_{\rho_1, \dots, \rho_b}^{\boldsymbol{\eta}}(\nu)$ where ν is an arbitrary probability measure on Σ . Now we consider $\rho_1, \dots, \rho_b \in \mathcal{X}_{+,1}$ such that the cavity equation has a unique solution $\Phi_{\rho_1, \dots, \rho_b}^{\boldsymbol{\eta}}$ together with an arbitrary $\bar{\rho}_1 \in \mathcal{X}_{+,1}$ and $\rho \in \mathcal{M}_{+,1}$. We use the shorter notations $R = R_{\rho_1, \dots, \rho_b}^{\boldsymbol{\eta}}$, $\bar{R} = R_{\bar{\rho}_1, \dots, \rho_b}^{\boldsymbol{\eta}}$ and $\Phi = \Phi_{\rho_1, \dots, \rho_b}^{\boldsymbol{\eta}}$. We have

$$\begin{aligned} \rho = \bar{R}(\rho) &\Leftrightarrow \rho - \Phi = \bar{R}(\rho) - R(\Phi) \\ &\Leftrightarrow \rho - \Phi = \bar{R}(\rho - \Phi) + \bar{R}(\Phi) - R(\Phi) \\ &\Leftrightarrow (I - \bar{R})(\rho - \Phi) = (\bar{R} - R)(\Phi). \end{aligned}$$

But $\rho - \Phi \in \mathcal{M}_0$, where $I - \bar{R}$ is invertible because of Proposition 5.7. Consequently there is a unique solution to the cavity equation with parameters $\bar{\rho}_1, \rho_2, \dots, \rho_b \in \mathcal{X}_{+,1}$ and it is given by formula (5.9). In the same way, when we change any other of the $\rho_i \in \mathcal{X}_{+,1}$ the solution to the cavity equation still exists and remains unique, so we can change all ρ_1, \dots, ρ_b to arbitrary elements of $\mathcal{X}_{+,1}$.

The monotonicity of the solution to the cavity equation follows at once from the fact that, for any probability measure $\nu \in \mathcal{M}_{1,+}$,

$$\Phi_{\rho_1, \dots, \rho_b}^{\boldsymbol{\eta}} = \lim_k (R_{\rho_1, \dots, \rho_b}^{\boldsymbol{\eta}})^k(\nu) \quad (5.10)$$

(according to the first point of Proposition 5.7) together with the monotonicity of $R_{\rho_1, \dots, \rho_b}^{\boldsymbol{\eta}}(\rho)$ in all parameters $\rho \in \mathcal{M}_{+,1}$, $\rho_1, \dots, \rho_b \in \mathcal{X}_{+,1}$ and $\boldsymbol{\eta}$ such that for all $i \in \{1, \dots, b\}$, $\rho_i^{\sigma_0}$ increases stochastically with the boundary condition $\sigma_0 \in \Sigma$. \square

5.3.4. Existence of a fixed point for the cavity equation.

We consider

$$\nu_n^{\boldsymbol{\eta}} = \mu_{\mathbb{T}_n}^{\boldsymbol{\eta},+}(\sigma_r \in \cdot) \quad (5.11)$$

for each $n \geq 0$ and $\boldsymbol{\eta} \in \Sigma$, where $\mu_{\mathbb{T}_n}^{\boldsymbol{\eta},+}$ is the quantum Gibbs measure with parameters $\beta, \mathbf{h}, \lambda$ on the tree \mathbb{T}_n , with plus boundary condition on the leaves and field $\boldsymbol{\eta}$ at the root. Observe that $\nu_0 = \delta_+$ and $\nu_{n+1} = \Phi_{\nu_n, \dots, \nu_n}$, the solution to the cavity equation with parameters ν_n, \dots, ν_n .

Lemma 5.11. *Let $\beta, \lambda, M > 0$. There is $c > 0$ such that, for all $\mathbf{h}, \boldsymbol{\eta}, \bar{\boldsymbol{\eta}}$ with L^1 -norm at most M and all $n \in \mathbb{N}^*$,*

(1) If $\bar{\eta} \geq \eta$,

$$(\nu_n^{\bar{\eta}} - \nu_n^{\eta})(\sigma \bullet \mathbf{1}) \geq c(\bar{\eta} - \eta) \bullet \mathbf{1}. \quad (5.12)$$

(2)

$$c \leq \frac{d\nu_n^{\eta}}{d\varphi} \leq c^{-1} \quad (5.13)$$

Proof. The first point is a consequence of Theorem 3.2 together with the DLR equation while the second one follows from the definition of $\mu_{\mathbb{T}_n}^{\eta,+}$ together with the DLR equation. \square

Proposition 5.12. *For every $\eta \in L^1([0, \beta])$, the stochastically decreasing sequence of probability distributions ν_n^{η} converge to a probability distribution ν_{∞}^{η} on Σ . The conditional distribution $\nu_{\infty} \in \mathcal{X}_{1,+}$ is a fixed point of the cavity equation, i.e. $\nu_{\infty} = \Phi_{\nu_{\infty}, \dots, \nu_{\infty}}$.*

Proof. The fact that the sequence ν_n^{η} is stochastically decreasing is an inductive consequence of the monotonicity of the solution to the cavity equation as $\nu_1 \prec \nu_0 = \delta_+$. Now consider the π -system

$$\Pi = \{ \{ \sigma(t) = +, \forall t \in I \}, I \subset [0, \beta] \text{ finite} \}.$$

Obviously $\sigma(\Pi)$ is the whole σ -algebra corresponding to the Skorohod topology. Furthermore, any $A \in \Pi$ is an increasing event and consequently $\nu_n^{\eta}(A)$ has a decreasing limit. On the other hand, there is $C < \infty$ such that the Radon-Nikodym derivative $\frac{d\nu_n^{\eta}}{d\varphi}$ is uniformly bounded by C , for any $n \geq 0$, and consequently (ν_n^{η}) is tight. Indeed, if Σ_k is the compact set of spin configurations with at most k flips, then $\nu_n^{\eta}(\Sigma_k^c) \leq C\varphi(\Sigma_k^c)$. This proves existence and uniqueness of the limit ν_{∞}^{η} . The conclusion that ν_{∞} is a fixed point of the cavity equation is an obvious consequence of the continuity of the solution to the cavity equation along its parameters, see (5.9) and (5.8). Note that if we are not in the uniqueness regime, the cavity equation has another fixed point corresponding to minus boundary condition. \square

5.3.5. Derivative of the solution of the cavity equation at the fixed point. Two natural norms on \mathcal{X} are

$$\|\rho\|_{\infty, \mathcal{X}} = \sup_{\eta} \|\rho^{\eta}\|_{\text{TV}} \quad (5.14)$$

$$\|\rho\|_{1, \mathcal{X}} = \int d\varphi(\eta) \|\rho^{\eta}\|_{\text{TV}}. \quad (5.15)$$

The first norm makes of \mathcal{X} a Banach space as \mathcal{M} itself is a Banach space. As far as the second norm is concerned, we observe that when ρ^{η} is absolutely continuous with respect to φ , $\|\rho\|_{1, \mathcal{X}} = \frac{1}{2} \int d\varphi(\eta) d\varphi(\sigma) \left| \frac{d\rho^{\eta}}{d\varphi} \right|$.

Now we introduce the derivative of the solution of the cavity equation $\Phi_{\rho_1, \dots, \rho_b}^{\eta}$ along the first variable ρ_1 in the direction $\rho \in \mathcal{X}_0$, at the fixed point $\rho_1 = \dots = \rho_b = \nu_{\infty}$. The formula follows at once from (5.9):

$$D^{\eta}(\rho) = (I - R_{\nu_{\infty}, \dots, \nu_{\infty}}^{\eta})^{-1} R_{\rho, \nu_{\infty}, \dots, \nu_{\infty}}^{\eta}(\nu_{\infty}^{\eta}), \forall \rho \in \mathcal{X}_0. \quad (5.16)$$

Note that, for all $\eta \in \Sigma$ and $\rho \in \mathcal{X}_0$, $D^{\eta}(\rho) \in \mathcal{M}_0$. Consequently $D(\rho) : \eta \mapsto D^{\eta}(\rho) \in \mathcal{X}_0$ and D can be seen as a linear operator on \mathcal{X}_0 . Our main theorem gives a bound on the norm of the operator norm of the k -th iterate of D , denoted by D^k , from the space $(\mathcal{X}_0, \|\cdot\|_{1, \mathcal{X}})$ to $(\mathcal{X}_0, \|\cdot\|_{\infty, \mathcal{X}})$:

Theorem 5.13. *There is $C < \infty$ that depends only on $\beta, \mathbf{h}, \lambda, b$ such that, for all $\rho \in \mathcal{X}_0$ and for all $k \geq 1$,*

$$\|D^k(\rho)\|_{\infty, \mathcal{X}} \leq \frac{C}{b^k} \|\rho\|_{1, \mathcal{X}}. \quad (5.17)$$

In particular, the spectral radius of D is at most $1/b$.

Remark 5.14. The Krein-Rutman theorem [3] states that the spectral radius of a strictly positive and compact operator is an eigenvalue corresponding to a positive eigenvector. Although we do not use this theorem, it is remarkable that in the proof below the asymptotic direction of the stochastically positive conditional measure $\nu_n - \nu_\infty$ helps to control the norm of D .

Proof. We first establish two preliminary results. Let $\rho \in \mathcal{X}_0$ and assume that ρ^{σ_0} is stochastically positive, for every $\sigma_0 \in \Sigma$. Then (i) in Proposition 5.9 together with (ii) in Proposition 5.7 imply that, for any $\eta \in \Sigma$,

$$\begin{aligned} \|D^\eta(\rho)\|_{\text{TV}} &\leq \frac{\Gamma}{1-\gamma} \int \nu_\infty^\eta(d\sigma_0) \rho^{\sigma_0}(\sigma \bullet \mathbf{1}) \\ &\leq \frac{C\Gamma}{1-\gamma} \int \varphi(d\sigma_0) \rho^{\sigma_0}(\sigma \bullet \mathbf{1}). \end{aligned} \quad (5.18)$$

where in the second inequality we have used (5.13). On the other hand, with the same assumption on ρ , if we take $f : \Sigma \rightarrow \mathbf{R}$ increasing with $f(+) = -f(-) = 1$, then

$$D^\eta(\rho)(f) \geq c \int \varphi(d\sigma_0) \rho^{\sigma_0}(\sigma \bullet \mathbf{1}) \quad (5.19)$$

for some $c > 0$ depending only on $\beta, \mathbf{h}, \lambda, b$. Indeed, each term in the expansion of $(I - R_{\nu_\infty, \dots, \nu_\infty}^\eta)^{-1}$ leaves the set of stochastically positive $\rho \in \mathcal{X}_0$ invariant. So

$$\begin{aligned} D^\eta(\rho)(f) &\geq R_{\rho, \nu_\infty, \dots, \nu_\infty}^\eta(\nu_\infty^\eta)(f) \\ &\geq 2c \int \nu_\infty^\eta(d\sigma_0) \rho^{\sigma_0}(\sigma \bullet \mathbf{1}) \end{aligned}$$

where at the second line we use (ii) in Proposition 5.9. Inequality (5.19) follows then from (5.13). Note that already inequalities (5.18) and (5.19) show that $\|D(\rho)\|_{\infty, \mathcal{X}} \leq C\|D(\rho)\|_{1, \mathcal{X}}$ when ρ^{σ_0} is stochastically positive, for every $\sigma_0 \in \Sigma$.

The derivative D is closely related to the way ν_n converges to its limit ν_∞ . As a consequence of (5.9) in Theorem 5.10,

$$\begin{aligned} \nu_{n+1}^\eta - \nu_\infty^\eta &= \Phi_{\nu_n, \dots, \nu_n}^\eta - \Phi_{\nu_\infty, \dots, \nu_\infty}^\eta \\ &= \Phi_{\nu_n, \dots, \nu_n}^\eta - \Phi_{\nu_n, \dots, \nu_n, \nu_\infty}^\eta + \\ &\quad \Phi_{\nu_n, \dots, \nu_n, \nu_\infty}^\eta - \Phi_{\nu_n, \dots, \nu_n, \nu_\infty, \nu_\infty}^\eta + \\ &\quad \vdots \\ &\quad \Phi_{\nu_n, \nu_\infty, \dots, \nu_\infty}^\eta - \Phi_{\nu_\infty, \dots, \nu_\infty}^\eta \\ &= (I - R_{\nu_n, \dots, \nu_n}^\eta)^{-1} R_{\nu_n - \nu_\infty, \nu_\infty, \dots, \nu_n}^\eta (\Phi_{\nu_n, \dots, \nu_n, \nu_\infty}^\eta) + \\ &\quad \vdots \\ &\quad (I - R_{\nu_n, \nu_\infty, \dots, \nu_\infty}^\eta)^{-1} R_{\nu_n - \nu_\infty, \nu_\infty, \dots, \nu_\infty}^\eta (\Phi_{\nu_\infty, \dots, \nu_\infty, \nu_\infty}^\eta) \\ &= bD^\eta(\nu_n - \nu_\infty) + r_n^\eta \end{aligned} \quad (5.20)$$

with $\|r_n^\eta\|_{\text{TV}} \leq C \|\nu_n - \nu_\infty\|_{\infty, \mathcal{X}}^2$ for some finite C , since $R_{\rho_1, \dots, \rho_b}^\eta(\rho)$ is multilinear and bounded, cf. (5.8). It follows by induction that, for every $k \geq 1$,

$$\nu_{n+k} - \nu_\infty = b^k D^k(\nu_n - \nu_\infty) + r_{k,n} \text{ as } n \rightarrow \infty, \quad (5.21)$$

where $\|r_{k,n}\|_{\infty, \mathcal{X}} \leq C_k \|\nu_n - \nu_\infty\|_{\infty, \mathcal{X}}^2$.

Now we consider $\rho \in \mathcal{X}_0$ such that $\rho^{\sigma_0} \succ 0$ for every $\sigma_0 \in \Sigma$. Without loss of generality we assume that $\|\rho\|_{1, \mathcal{X}} = 1$. From (5.20) we know that $\nu_n^\eta - \nu_\infty^\eta$ is close to the image by bD^η of $\nu_{n-1}^\eta - \nu_\infty^\eta$ as $n \rightarrow \infty$. According to (5.18) and (5.19) this implies the existence of $c > 0$ such that, for all n large enough, for every f increasing with $f(\pm 1) = \pm 1$,

$$\inf_{\eta} (\nu_n^\eta - \nu_\infty^\eta)(f) \geq c \sup_{\eta} \|\nu_n^\eta - \nu_\infty^\eta\|_{\text{TV}} = c \|\nu_n - \nu_\infty\|_{\infty, \mathcal{X}}.$$

From (5.18) and the assumption that $\|\rho\|_{1, \mathcal{X}} = 1$ it follows that

$$\sup_{\eta} \|D^\eta(\rho)\|_{\text{TV}} \leq C' = \frac{C\Gamma}{1-\gamma}$$

According to the last two displays, for every n large enough, for all $\eta \in \Sigma$,

$$\frac{c}{2C'} D^\eta(\rho) \prec \frac{\nu_n^\eta - \nu_\infty^\eta}{\|\nu_n - \nu_\infty\|_{\infty, \mathcal{X}}}.$$

Call $c' = c/(2C')$. As D preserves stochastic positivity,

$$D^{k-2} \left(\frac{\nu_n - \nu_\infty}{\|\nu_n - \nu_\infty\|_{\infty, \mathcal{X}}} - c' D(\rho) \right) = \frac{1}{b^{k-2}} \frac{\nu_{n+k-2} - \nu_\infty - r_{k-2,n}}{\|\nu_n - \nu_\infty\|_{\infty, \mathcal{X}}} - c' D^{k-1}(\rho)$$

is also stochastically positive ($r_{k-2,n}$ was defined at (5.20)). It follows that, for all $\eta \in \Sigma$,

$$\left(D^{k-1} \right)^\eta(\rho) (\sigma \bullet \mathbf{1}) \leq \frac{1}{c' b^{k-2}} \frac{\|\nu_{n+k-2} - \nu_\infty\|_{\infty, \mathcal{X}} + \|r_{k-2,n}\|_{\infty, \mathcal{X}}}{\|\nu_n - \nu_\infty\|_{\infty, \mathcal{X}}}.$$

for any n large enough. We can obviously find a subsequence of n along which $\|\nu_{n+k-2} - \nu_\infty\|_{\infty, \mathcal{X}} \leq \|\nu_n - \nu_\infty\|_{\infty, \mathcal{X}}$, so taking \liminf_n shows that

$$\left(D^{k-1} \right)^\eta(\rho) (\sigma \bullet \mathbf{1}) \leq \frac{1}{c' b^{k-2}}.$$

According to (5.18) we have

$$\left\| D^k(\rho) \right\|_{\infty, \mathcal{X}} \leq \sup_{\eta \in \Sigma} \frac{C\Gamma}{1-\gamma} \int \varphi(d\sigma_0) \left(D^{k-1} \right)^{\sigma_0}(\rho) (\sigma \bullet \mathbf{1})$$

and therefore, for some different $C < \infty$

$$\left\| D^k(\rho) \right\|_{\infty, \mathcal{X}} \leq \frac{C}{b^k} \quad (5.22)$$

for all $k \geq 1$, all $\rho \in \mathcal{X}_0$ with $\|\rho\|_{1, \mathcal{X}} = 1$, such that $\rho^{\sigma_0} \succ 0$ for every $\sigma_0 \in \Sigma$. Finally we extend (5.22) to any $\rho \in \mathcal{X}_0$ with $\|\rho\|_{1, \mathcal{X}} = 1$. Consider

$$\begin{aligned} \rho_1^{\sigma_0} &= \|\rho^{\sigma_0}\|_{\text{TV}} (\delta_+ - \delta_-) + \rho^{\sigma_0} \\ \rho_2^{\sigma_0} &= \|\rho^{\sigma_0}\|_{\text{TV}} (\delta_+ - \delta_-) - \rho^{\sigma_0}. \end{aligned}$$

Note that both $\rho_1^{\sigma_0}$ and $\rho_2^{\sigma_0}$ are stochastically positive for any $\sigma_0 \in \Sigma$. For instance, if $f : \Sigma \rightarrow \mathbf{R}$ is increasing, then $\rho_1^{\sigma_0}(f) = (f(+)-f(-)) \|\rho^{\sigma_0}\|_{\text{TV}} + \rho^{\sigma_0}(f) \geq 0$. According to the triangular inequality (note that $\|\rho_1\|_{1,\mathcal{X}} \leq 2$) and to (5.22),

$$\|D^k(\rho)\|_{\mathcal{X}}^{\infty} \leq \frac{4C}{b^k}$$

and we are done. \square

6. PROOF OF THE MAIN THEOREMS

6.1. Proof of Theorem 5.3. Without loss of generality, we only consider here the case $x = 0$ in the definition of κ . So we fix some $l \geq 0$ and $z \in \mathbb{T}_l^b$. We denote by $k = |z|$ the depth of z . Now we consider the subtree of \mathbb{T}_l^b issued from r that contains z , and root it at z . This is a regular tree with $b+1$ children at z and b children otherwise (except on the leaves). Its depth is not uniform and ranges between $\min(l-k, k)$ and $l+k$. The boundary condition is uniformly plus, except on the leaf r where it is $+$ or $-$.

Let $(z_0 = r, z_1, \dots, z_k = z)$ be a path from r to z . For any $i \in \{1, \dots, k-1\}$, we call $\nu_i^{\pm, \xi}$ the conditional distribution $\mu(\sigma_{z_i} \in \cdot | \sigma_{z_{i+1}} = \xi, \sigma_r = \pm)$ and $\rho_{(i,j)}^{\xi} = \mu(\sigma_{y_j} \in \cdot | \sigma_{z_{i+1}} = \xi)$ where y_j is the j -th children of z_{i+1} for $j = 2, \dots, b$, if we consider z_i as its first children. We have,

$$\nu_i^{\pm} = \Phi_{\nu_{i-1}^{\pm}, \rho_{(i-1,2)}, \dots, \rho_{(i-1,b)}}, \quad i \leq k-1$$

and also, if we call $\nu_k^{\pm, 0}$ the marginal distribution $\mu(\sigma_z \in \cdot | \sigma_r = \pm)$ and $\rho_j^{\xi} = \mu(\sigma_{y_j} \in \cdot | \sigma_z = \xi)$ where y_j is the j -th children of z for $j = 2, \dots, b+1$, apart from z_1 , then

$$\nu_k^{\pm, 0} = \Phi_{\nu_{k-1}^{\pm}, \rho_2, \dots, \rho_{b+1}}$$

(note the $b+1$ parameters in the cavity equation).

Now we claim that, given a neighborhood $V \subset \mathcal{X}_{1,+}$ of ν_{∞} in the $\|\cdot\|_{\infty, \mathcal{X}}$ norm, for all i such that $\min(i, l-i)$ is large enough depending on V , every $\rho_{(i,j)}$ and ν_i^{\pm} lies in V . This is clear for $\rho_{(i,j)}$ and ν_i^+ as it is the marginal distribution at the root of a large subtree with identical plus boundary condition. As $\|\nu_{i+1}^+ - \nu_{i+1}^-\|_{\infty, \mathcal{X}} \leq \gamma \|\nu_i^+ - \nu_i^-\|_{\infty, \mathcal{X}}$ where $\gamma < 1$ as been defined in (5.2), the same holds for ν_i^- .

According to (5.9), for all $\varepsilon > 0$, for all i with $\min(i, l-i)$ large enough depending on ε but not on k, l ,

$$\|\nu_{i+1}^+ - \nu_{i+1}^- - D(\nu_i^+ - \nu_i^-)\|_{\infty, \mathcal{X}} \leq \varepsilon \|\nu_i^+ - \nu_i^-\|_{\infty, \mathcal{X}}$$

where D , the derivative of the solution of the cavity equation, at the fixed point ν_{∞} , along the first parameter, has been defined at (5.16). This clearly extends as follows. Fix $j \in \mathbf{N}^*$ and $\varepsilon > 0$. Then, there is $a = a(\varepsilon, j)$ that does not depend on k, l such that, for all i with $\min(i, l-i) \geq a$, then

$$\|\nu_{i+j}^+ - \nu_{i+j}^- - D^j(\nu_i^+ - \nu_i^-)\|_{\infty, \mathcal{X}} \leq \varepsilon \|\nu_i^+ - \nu_i^-\|_{\infty, \mathcal{X}}.$$

According to Theorem 5.13 we conclude that, under the same conditions,

$$\|\nu_{i+j}^+ - \nu_{i+j}^-\|_{\infty, \mathcal{X}} \leq \left(\frac{C}{b^j} + \varepsilon\right) \|\nu_i^+ - \nu_i^-\|_{\infty, \mathcal{X}}$$

where $C < \infty$ depends only on $\beta, \mathbf{h}, \lambda$. From $\|\nu_{i+1}^+ - \nu_{i+1}^-\|_{\infty, \mathcal{X}} \leq \gamma \|\nu_i^+ - \nu_i^-\|_{\infty, \mathcal{X}}$ it follows that

$$\left\| \nu_k^{+,0} - \nu_k^{-,0} \right\|_{\text{TV}} \leq \left(\frac{C}{b^j} + \varepsilon \right)^{[(k-2a)/j]}$$

which proves the first statement of the Theorem.

Now we consider $\beta, \mathbf{h}, \lambda$ in the uniqueness regime. Let $\tau \in \Sigma^{\mathbb{T}^b}$ and denote μ_l^τ the Gibbs measure on \mathbb{T}_l^b with τ acting as a boundary condition on the leaves. Similarly to the proof of the first point of Proposition 5.9, we have, for any $y, z \in \mathbb{T}_l^b$ with y the ancestor of z , and every $\eta \in \Sigma$,

$$\begin{aligned} \left\| \mu_l^+ (\sigma_z \in \cdot | \sigma_y = \eta) - \mu_l^\tau (\sigma_z \in \cdot | \sigma_y = \eta) \right\|_{\text{TV}} &\leq \\ \Gamma \sum_{x \text{ child of } z} \left\| \mu_l^+ (\sigma_x \in \cdot | \sigma_y = \eta) - \mu_l^\tau (\sigma_x \in \cdot | \sigma_y = \eta) \right\|_{\text{TV}}. \end{aligned}$$

This proves that, as before, every $\rho_{(i,j)}$ and ν_i^\pm lies in a given neighborhood $V \subset \mathcal{X}_{1,+}$ of ν_∞ in the $\|\cdot\|_{\infty, \mathcal{X}}$ norm, for all i such that $\min(i, l-i)$ is large enough depending on V but not on τ . The rest of the argument is identical.

6.2. Proof of Theorem 5.4. Let $\tau \in \Sigma^{\mathbb{T}^b}$ be such that $b\gamma\kappa(\tau) < 1$. We recall that, from Theorem 5.3, the plus boundary condition always satisfies this requirement, and that, if $\beta, \mathbf{h}, \lambda$ are in the uniqueness region, we could choose any τ .

6.2.1. Spectral gap. Following [7] (see Section 3 and the proof of Theorem 4.3), for the proof of the first part of Theorem 5.4 it is enough to verify (see Equation (14) in [7]) that for all $\varepsilon > 0$, for all $k \geq 1$ large enough depending on ε , for all $l \geq k$, for any $g \in L^2(\Sigma, d\varphi)$ and all $\eta \in \Sigma$,

$$\text{Var}_{\mu_l} \left(\mu_l \left(g(\sigma_r) | \{\sigma_z\}_{|z|=k} \right) \right) \leq ((1+\varepsilon)\kappa\gamma b)^{k/2} \text{Var}_{\mu_l} (g(\sigma_r)) \quad (6.1)$$

where μ_l is the Gibbs measure on \mathbb{T}_l^b with boundary condition τ on the leaves and extra field η acting on the root. In what follows, all the bounds will be uniform in η .

Let us denote $K(\sigma_r, \cdot)$ the Radon-Nikodym derivative of the measure $\mu_l \left(\mu'_r \in \cdot | \{\sigma_z\}_{|z|=k} \right) | \sigma_r$ with respect to the marginal distribution of the root $\nu = \mu_l(\sigma_r \in \cdot)$. The kernel K is constructed as follows: given σ_r sample the spins at distance k from the root. Then take these spins and η acting on top of the root as a boundary condition on \mathbb{T}_k^b and sample again σ'_r . It is useful to remark that K is uniformly bounded. The Cauchy-Schwarz inequality implies that

$$\begin{aligned} \text{Var}_{\mu_l} \left(\mu_l \left(g(\sigma_r) | \{\sigma_z\}_{|z|=k} \right) \right) &= \text{Cov}_{\mu_l} \left(\mu_l \left(\mu_l \left(g(\sigma_r) | \{\sigma_z\}_{|z|=k} \right) | \sigma_r \right), g(\sigma_r) \right) \\ &\leq \text{Var}_{\mu_l} \left(\mu_l \left(g(\sigma_r) | \{\sigma_z\}_{|z|=k} \right) \right)^{1/2} \text{Var}_{\mu_l} (g(\sigma_r))^{1/2}. \end{aligned}$$

Then

$$\begin{aligned}
& \text{Var}_{\mu_l} \left(\mu_l \left(g(\sigma_r) \mid \{\sigma_z\}_{|z|=k} \right) \right) \\
&= \text{Var}_{\mu_l} \left(\int K(\sigma_r, \sigma'_r) g(\sigma'_r) d\nu(\sigma'_r) \right) \\
&= \frac{1}{2} \nu \otimes \nu \left(\left(\int K(\sigma_r^1, \sigma'_r) g(\sigma'_r) d\nu(\sigma'_r) - \int K(\sigma_r^2, \sigma'_r) g(\sigma'_r) d\nu(\sigma'_r) \right)^2 \right) \\
&\leq \frac{1}{2} \nu \otimes \nu \left(\int (K(\sigma_r^1, \sigma'_r) - K(\sigma_r^2, \sigma'_r))^2 d\nu(\sigma'_r) \times \int (g(\sigma'_r) - \mu(g))^2 d\nu(\sigma'_r) \right) \\
&= \frac{1}{2} \text{Var}_{\mu_l}(g) \times \nu \otimes \nu \otimes \nu \left[(K(\sigma_r^1, \sigma_r^3) - K(\sigma_r^2, \sigma_r^3))^2 \right]
\end{aligned}$$

so (6.1) would follow from

$$\nu \otimes \nu \otimes \nu \left[(K(\sigma_r^1, \sigma_r^3) - K(\sigma_r^2, \sigma_r^3))^2 \right] \leq 2((1 + \varepsilon) \kappa \gamma b)^k$$

From the boundedness of K we conclude that, for some constant C ,

$$\begin{aligned}
& \nu \otimes \nu \otimes \nu \left[(K(\sigma_r^1, \sigma_r^3) - K(\sigma_r^2, \sigma_r^3))^2 \right] \\
&\leq 4C \sup_{\sigma_r^1, \sigma_r^2 \in \Sigma} \|K(\sigma_r^1, \cdot) \nu - K(\sigma_r^2, \cdot) \nu\|_{\text{TV}} \\
&\leq 8C \sup_{\rho \in \Sigma} \|K(+, \cdot) \nu - K(\rho, \cdot) \nu\|_{\text{TV}}.
\end{aligned}$$

The DLR property, together with the definition of Γ at (3.2), imply that for ordered marginals the TV distance is comparable with difference of expectation of $\sigma \bullet \mathbf{1}$. Consequently,

$$\begin{aligned}
& \|K(+, \cdot) \nu - K(\rho, \cdot) \nu\|_{\text{TV}} \\
&\leq C \left[\mu_l \left(\mu_l(\sigma'_r \bullet \mathbf{1} \mid \{\sigma_z\}_{|z|=k}) \mid \sigma_r = + \right) - \mu_l \left(\mu_l(\sigma'_r \bullet \mathbf{1} \mid \{\sigma_z\}_{|z|=k}) \mid \sigma_r = \rho \right) \right]
\end{aligned}$$

which is clearly maximum if we take $\rho = -$. Now, we argue that the former difference is bounded by

$$\left[\sum_{z: |z|=k} \mu_l(\sigma_z \bullet \mathbf{1} \mid \sigma_r = +) - \mu_l(\sigma_z \bullet \mathbf{1} \mid \sigma_r = -) \right] \times 2\Gamma \gamma^{k-1} \quad (6.2)$$

For proving this we need only a slight adaptation of the proof of (ii) in Claim 4.4 in [7]. Consider two spin configurations ξ, ξ' that differ at a single position z with $|z| = k$. We can easily construct a coupling of $\mu_l(\sigma_r \in \cdot \mid \{\sigma_z = \xi'_z\}_{|z|=k})$ and $\mu_l(\sigma_r \in \cdot \mid \{\sigma_z = \xi_z\}_{|z|=k})$ for which the two variables differ with probability at most $\gamma^{k-1} \Gamma \|\xi' - \xi\|_1$ according to the definition of Γ in Proposition 3.1 and to that of γ at (5.2). Now we consider a monotone coupling of $\{\sigma_z\}_{|z|=k}$ corresponding to the conditions $\sigma_r = \pm$. By applying the former coupling to an interpolating sequence between these spin configurations as in proof of (ii) in Claim 4.4 in [7], we conclude the proof of (6.2).

Finally, $\mu_l(\sigma_z \bullet \mathbf{1} \mid \sigma_r = +) - \mu_l(\sigma_z \bullet \mathbf{1} \mid \sigma_r = -) \leq (1 + \varepsilon)^k \kappa^k$ for all $k \geq 0$ large enough according to the definition of κ . The proof of the first part of Theorem 5.4 is complete.

6.2.2. Mixing time. We first establish an intermediate step. We fix l and consider the Gibbs measure on \mathbb{T}_l^b with boundary condition τ on the leaves of \mathbb{T}_l^b and additional field $\eta \in \Sigma$ at the root. Now we turn to the dynamics. Consider as a starting configuration the identically plus configuration. Similarly we could consider the identically minus configuration. Let P_t^+ be the distribution of the spin configuration at time t for the dynamics. We denote by $h_r^\eta(t)$ the Radon-Nikodym derivative of P_t^+ with respect to μ . Note that $h_r^\eta(0) = \mathbf{1}_{\{+\}}/\mu(+)$ and $h_r^\eta(t) = P_t h_r^\eta(0)$ because P_t is self-adjoint in $L^2(\mu)$.

We also define $T_r = \sup_\eta \min\{t : \text{Var}(h_r^\eta(t)) \leq 1\}$. Similarly, given $x \in \mathbb{T}_l^b \setminus \{r\}$, we consider the dynamics censored everywhere except on the subtree rooted at x and call $h_x^+(t)$ the resulting Radon-Nikodym derivative with respect to the Gibbs measure conditioned on being plus outside the subtree of x . We then define $T_x = \min\{t : \text{Var}(h_x^+(t)) \leq 1\}$.

Now we prove that there exists a constant t_0 independent of l such that

$$T_x \leq \max_{y \text{ child of } x} T_y + t_0. \quad (6.3)$$

For this purpose we use censoring (Proposition 4.6) together with the first point of Theorem 5.4. For simplicity we only consider the case $x = r$. We censor for time $t = \max_{y \text{ child of } r} T_y$ the root and then run the uncensored dynamics for an extra time t_0 to be determined later. Let $\bar{h}_r^\eta(t)$ be the Radon-Nikodym derivative of the corresponding distribution at time t . Then,

$$\begin{aligned} \text{Var}(h_r^\eta(t + t_0)) &\leq \text{Var}(\bar{h}_r^\eta(t + t_0)) \\ &= \text{Var}(P_{t_0} \bar{h}_r^\eta(t)) \\ &\leq e^{-2 \text{gap} \times t_0} \text{Var}(\bar{h}_r^\eta(t)). \end{aligned}$$

Now we prove that $\text{Var}(\bar{h}_r^\eta(t))$ is bounded uniformly in l . By construction we have

$$\bar{h}_r^\eta(t)(\sigma) = \frac{\mathbf{1}_{\{+\}}(\sigma_r)}{\mu(\sigma_r = +)} \prod_{y \text{ child of } r} h_y^+(t)(\sigma_{\mathbb{T}_y})$$

where $\sigma_{\mathbb{T}_y}$ is the restriction of σ to the subtree rooted at y . Finally, we remark that

$$\begin{aligned} \text{Var}(\bar{h}_r^\eta(t)) &\leq \mu\left((\bar{h}_r^\eta(t))^2\right) \\ &= \mu(\sigma_r = +) \mu\left((\bar{h}_r^\eta(t))^2 \mid \sigma_r = +\right) \\ &= \frac{1}{\mu(\sigma_r = +)} \prod_{y \text{ child of } r} \mu\left((h_y^+(t)(\sigma_{\mathbb{T}_y}))^2 \mid \sigma_r = +\right) \\ &= \frac{1}{\mu(\sigma_r = +)} \prod_{y \text{ child of } r} [\text{Var}_{\mu(\cdot \mid \sigma_r = +)}(h_y^+(t)(\sigma_{\mathbb{T}_y})) + 1] \end{aligned}$$

which is smaller than $2^b/\mu(\sigma_r = +)$ according to the definition of t . If we take t_0 such that

$$\frac{2^b \exp(-2 \text{gap} \times t_0)}{\mu(\sigma_r = +)} \leq 1$$

then t_0 is bounded uniformly in l and we are done. In conclusion, we have shown that $T_r \leq lt_0$.

Remark 6.1. We observe that the recursive inequality (6.3) is analogous to the one obtained in [7] (see Lemma 5.8 there) for the logarithmic Sobolev constant. In our context, this constant is easily seen to be infinite because of configurations with an arbitrary large number of flips.

We now have to consider the dynamics starting from an arbitrary spin configuration ξ . Choose $t = T_r + cl$ for some $c > 0$ to be chosen later on. We have of course

$$\|P_t^\xi - \mu\|_{\text{TV}} \leq \|P_t^\xi - P_t^+\|_{\text{TV}} + \|P_t^+ - \mu\|_{\text{TV}}.$$

On one hand,

$$\|P_t^+ - \mu\|_{\text{TV}} = \|h_r^\eta(t) - 1\|_{L^1(\mu)} \leq \|h_r^\eta(t) - 1\|_{L^2(\mu)} \leq e^{-\text{gap}(t-T_r)} \leq e^{-\text{gap} \times cl} \quad (6.4)$$

according to the definition of T_r .

For the remaining part $\|P_t^\xi - P_t^+\|_{\text{TV}}$ we use a coupling argument. Call $s = T_r + cl/2$. We consider Ψ_s a monotone coupling of $P_s^\xi \prec P_s^+$ and denote (σ, σ^+) its variables. According to Markov's inequality, we have

$$\begin{aligned} \Psi_s(\exists x, \sigma_x \bullet 1 \leq \sigma_x^+ \bullet 1 - \varepsilon) &\leq \sum_x \Psi_s(\sigma_x \bullet 1 \leq \sigma_x^+ \bullet 1 - \varepsilon) \\ &\leq \varepsilon^{-1} \sum_x P_s^+(\sigma_x \bullet 1) - P_s^\xi(\sigma_x \bullet 1) \\ &\leq \varepsilon^{-1} \sum_x P_s^+(\sigma_x \bullet 1) - P_s^-(\sigma_x \bullet 1) \\ &\leq \frac{2b^l \beta}{\varepsilon} (\|P_s^+ - \mu\|_{\text{TV}} + \|P_s^- - \mu\|_{\text{TV}}). \end{aligned}$$

Now we take $\varepsilon = b^{-2l}$. According to the last display and to (6.4) the probability $\Psi_s(\exists x, \sigma_x \bullet 1 \leq \sigma_x^+ \bullet 1 - \varepsilon)$ can be made arbitrary small by taking c large. So with high probability under Ψ_s , at not position the spin configurations differ by more than ε in $L^1([0, \beta])$ distance at time $s = T_r + cl/2 = t - cl/2$. In the remaining time $cl/2$, we update the spin configurations at the same positions according to an optimal coupling of the marginals. Using Proposition 3.1 we see that, as long as the spin configurations differ at every x by at most ε in $L^1([0, \beta])$ distance, each update put locally the same spin with probability at least $1 - \Gamma\varepsilon$. Therefore we conclude that $\|P_t^\xi - P_t^+\|_{\text{TV}}$ can be made arbitrarily small if c is large enough. This concludes the proof that the mixing time is bounded by Cl .

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